Pitch-Symmetric Tetrachord Partitions

Polytrope

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This article outlines the underlying pitch-class structure of an original technique of twelve-tone musical composition which I frequently employ. I call this structure the system of *pitch-symmetric tetrachord partitions (PSTP).* I use PSTPs because I consider that they afford systematically a sense of harmonic progression while retaining the essential atonality[1] of twelve-tone music.

[1]I.e. lack of tonal centres or with free and rapid shifting of tonal centres.

1 Introduction

It is not unreasonable to say that in serial twelve-tone music the classical role of *keys* and *modes* in support of unity and structural organization is played instead by *tone rows,* or more properly speaking, *tone-row complexes*—each comprising a 'prime' or 'original' row form and other row forms derived from the prime form by combinations of transposition, pitch inversion, and order reversal. Spelling this out a bit, each note in a tonal piece relates to a prevailing harmonic function (either as a chord factor or as a departure therefrom leading in a more or less expected way onward), and each harmonic function ('dominant', 'tonic', etc) has a more or less specific role in a prevailing key and mode.[2] In a serial twelve-tone piece, on the other hand, each note belongs to a prevailing row form,[3] and often significantly to a motivic subsegment of that row form.

In the PSTP system, each note in a piece belongs to (or is a temporary departure from[4]) a prevailing four-pitch-class set[5] or *tetrachord,* and each tetrachord is a member of a prevailing set of three tetrachords whose union comprises all twelve pitch classes; that is, each tetrachord is a member of a prevailing *tetrachordal partition* of the twelve. Notes from the prevailing tetrachord may appear in any order and may reoccur arbitrarily often and in any registers: it is the *set* of four pitch-classes which is prevailing, not any particular pitch-class sequence.[6] The prevailing tetrachordal partition may change fairly frequently during the course of a PSTP piece, just as the prevailing key and mode in a tonal piece change fairly frequently at one level or another due to modulations, secondary dominant formations, etc.

2 Pitch-Symmetric Tetrachords

One further specification completes an outline of the PSTP system: Each of the tetrachords of a PSTP must be *inversionally pitch-symmetric (PS).* That is, it must be a set like

[2]A given function or a given note may of course have an ambiguous role, for example in pivot modulations, chord 'reinterpretations', etc.

[3]Occasionally more than one, for example when successive row forms are overlapped.

[4]Preferably stepwise.

[5]In this article, "set" denotes a collection considered without regard to any particular order of its elements. This is the usual mathematical sense of the word, as opposed to a sense sometimes used in music theory, according to which "set" is equivalent to "row".

[6]A simple sequential or simultaneous appearance of the four pitch classes is of course permitted.

{A, C, E, G}, in which A is nominally as far below C (three semitones) as G is nominally above E; or like

{C, D, E, G♯}, in which C is nominally as far below D (a whole tone) as E is nominally above it, and G♯ is a tritone away from D, so equally distant from C and E.

There are 15 types of PS tetrachords, as enumerated in Table 1. These and their enumeration are further described in another article on this website: "Pitch-Symmetric Tetrachords, etc." The numbers in the first three columns of Table 1 indicate the spacing of the constituent pitch classes of each tetrachord as numbers of semitones nominally above an arbitrary origin (0). This is construed musically in the "relative content" column. The numbers in the "content" column for a given tetrachord are arranged in circle-of-fifths order[7] to emphasize that aspect of the tetrachord's pitch-symmetry. For example, the entry "{0, 2, 9, 11}" for the *V* tetrachord corresponds to 0, 2, 3, and 5 perfect fifths above 0; i.e. two, then one more, then two more: '2-1-2'. The "name" column of Table 1 assigns a fixed capital letter for each of the 15 PS tetrachord types. In print, these should be italicized to minimize confusion with pitch-class names and harmonic function symbols.

3 Pitch-Symmetric Tetrachord Partitions

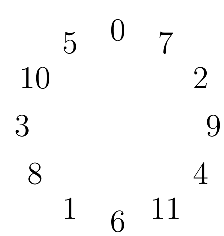

Figure 1: Circle-of-fifths order for Figures.

Figures 2-16 exhibit all the types of PSTPs, each figure showing all those that include instance(s) of a particular PS tetrachord type. The dots in each circle in those figures correspond to the twelve pitch classes in circle-of-fifths-order, as shown

[7]Cf. Figure 1.

content*	name*	acoustic root*	relative content	guide tone
{0, 7, 2, 9}	$0Q$	0	{M2, P5, M6}	
		7	{M2, P4, P5}	
		2	{P4, P5, m7}	
{0, 7, 9, 4}	$9N$	9	{m3, P5, m7}	
		0	{M3, P5, M6}	
{0, 7, 4, 11}	$0M$	0	{M3, P5, M7}	
		4	{m3, P5, m6}	
{0, 7, 11, 6}	$0T$	0	{A4, P5, M7}	A4
		11	{m2, P5, m6}	P5
{0, 7, 6, 1}	$0X$, $6X$	0, 6	{A1, A4, P5}	P5
{0, 2, 9, 11}	$2V$	2	{P5, M6, m7}	
{0, 2, 4, 6}	$0W$	0	{M2, M3, A4}	A4
		2	{M2, M3, m7}	M3
{0, 2, 11, 1}	$11S$	11	{m2, M2, m3}	
{0, 2, 6, 8}	$2F$, $8F$	2, 8	{M3, d5, m7}	M3
{0, 9, 4, 1}	$9L$	9	{m3, M3, P5}	
{0, 9, 11, 8}	$8K$	8	{m2, m3, d4}	
{0, 9, 6, 3}	$0D$, $3D$, $6D$, $9D$	0, 3, 6, 9	{m3, d5, d7}	m3
{0, 4, 11, 3}	$4U$	4	{P5, m6, M7}	
{0, 7, 2, 1}	$0R$	0	{m2, M2, P5}	P5
		7	{P4, d5, P5}	d5
{0, 2, 4, 8}	$8A$	0	{M2, M3, m6}	m6
		4	{d4, m6, m7}	d4
		8	{M3, A4, A5}	A4

* see text for interpretation of numbers.
acoustic root: nominally lower pitch class of a P5, M3, or m3 interval within the tetrachord.
relative content: relative to the indicated acoustic root as P1.
guide tone: relative to the indicated acoustic root as P1—see Hindemith, *The Craft of Musical Composition,* ch.III.10, IV.5 *et al.*

Table 1: PS tetrachord types, showing some musical characteristics.

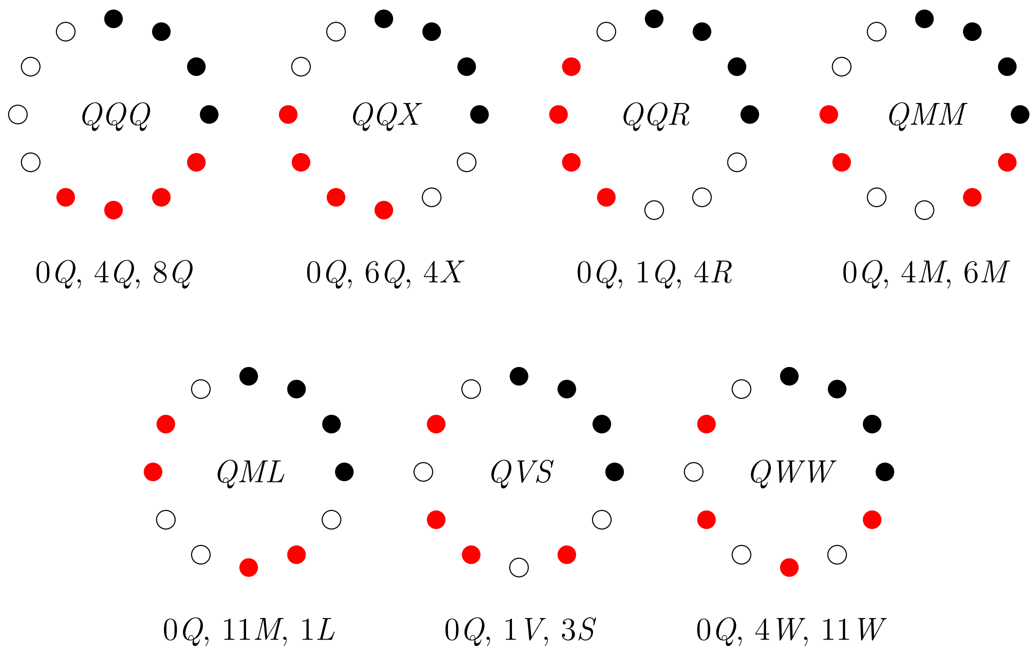

Figure 2: PSTP types with type Q tetrachords.

in Figure 1. In each circle the pitch classes of the particular tetrachord type which is the subject of the figure are indicated in black, those of the other two constituent tetrachord types in red and white. Internal sub-captions (like "$0Q$, $11M$, $1L$" in Figure 2) name the constituent tetrachord types in the order "black, red, white" with the prefixed numbers (e.g. "0, 11, 1") indicating the pitch-class relations which obtain among the constituent tetrachords in any particular PSTP of this type.

The label in the centre of each circle is either a) a combination of the names of the constituent tetrachord types, or b) in square brackets, a label of type (a) followed by '$+n$', where n is an integer from 1 to 11. A label of type (a) is used for the first occurrence of a particular PSTP type in Figures 2-16, and serves as a name for the PSTP type. A label of type (b) is used for subsequent occurrences of a particular PSTP type (under the headings of its other constituent PS tetrachords). The '$+n$' term in this case indicates the change in numbering from the original.

An example may make this clearer. The PSTP type labeled *"QML"* in Figure 2 is the set of three PS tetrachord types

$$\{\{0,7,2,9\},\{11,6,3,10\},\{4,1,8,5\}\} = \{0Q, 11M, 1L\}.$$

The same PSTP type appears in Figure 4 with the label "[QML+1]", where it is shown as the set

$$\{\{0,7,4,11\},\{1,8,3,10\},\{5,2,9,6\}\} = \{0M, 1Q, 2L\}.$$

A particular PSTP of this type is

$$\{\{\text{D, A, E, B}\},\{\text{D}\flat\text{, F, A}\flat\text{, C}\},\{\text{E}\flat\text{, G}\flat\text{, G}\natural\text{, B}\flat\}\} = \{\text{D}Q\text{, D}\flat M\text{, E}\flat L\}.$$

To match this with Figure 2, take 0 = D; to match it with Figure 4, take 1 = D. The difference accounts for the "+1" in the Figure 4 label. Note that this addition is *modulo* 12, so that e.g. $11 + 1 = 0$.

It turns out that there are 47 distinct PSTP types (those with labels of type (a) in Figures 2-16). Tables 2 and 3 list these again, showing the pitch-class relations among their constituent PS tetrachord types.

I can't see that the brute combinatorial facts—15 PS tetrachord types, 47 PSTP types—have any mathematical or musical significance. The truth of the former fact is argued in "Pitch-Symmetric Tetrachords, etc" (on this website). The latter fact can be ascertained through a computer enumeration, a program for which is included in Appendix B.

The PSTPs are somewhat analogous to Josef Matthias Hauer's 'tropes',[8] which comprise the hexachordal partitions of the 12 pitch classes. However, only pitch-symmetric tetrachords are components of the PSTPs, whereas all hexachords are included as components of the tropes. This, together with 1) the intrinsic difference between three-fold and two-fold partitions and 2) Hauer's special treatment of semitone-adjacent pitch classes, makes the PSTP and trope systems very different as compositional tools.

[8]See `www.musiker.at/sengstschmidjohann/stichwort-trope.php3` and "Hauer's Tropes and the Enumeration of Twelve-Tone Hexachords" on this website.

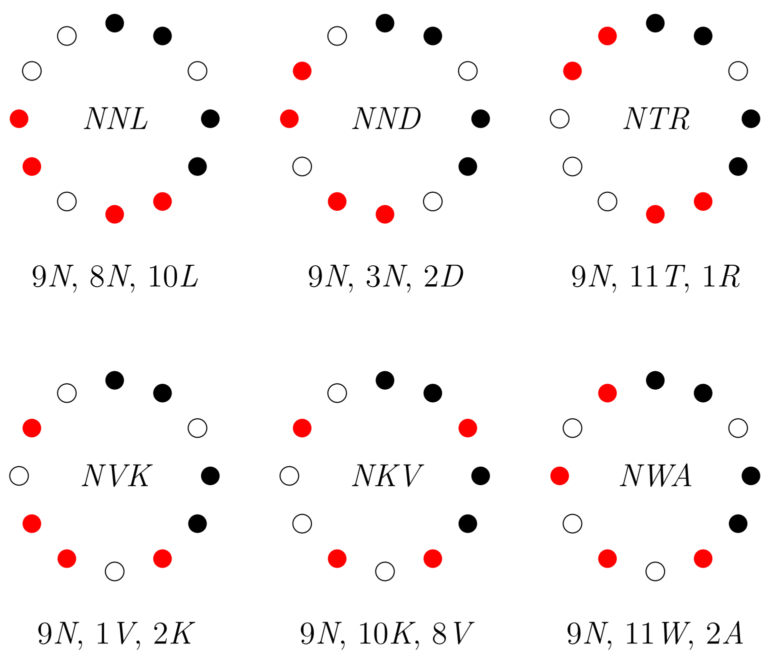

Figure 3: PSTP types with type N tetrachords.

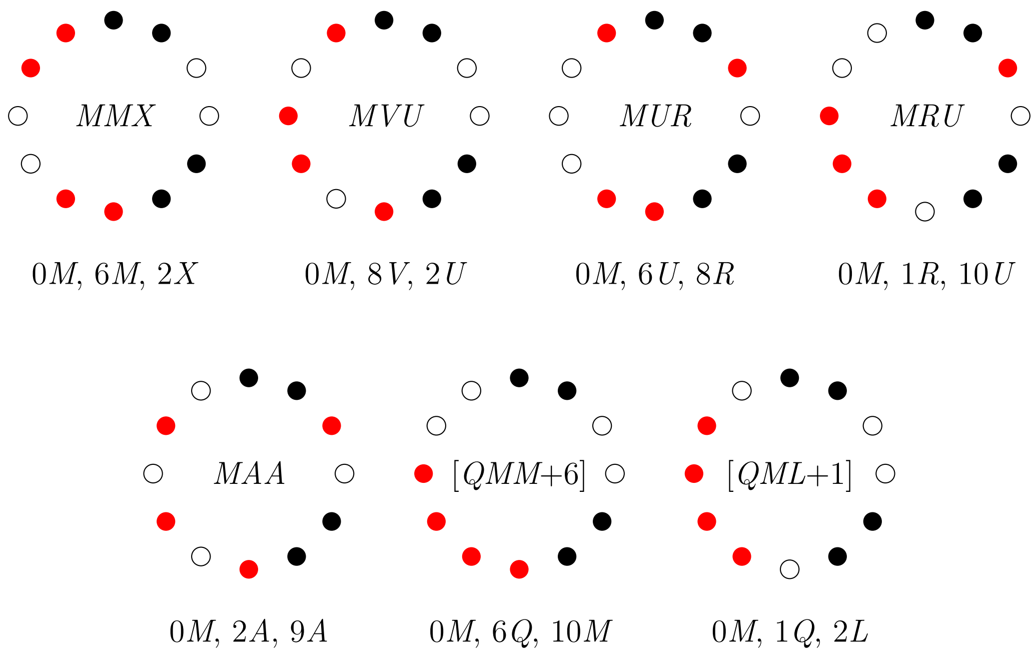

Figure 4: PSTP types with type M tetrachords.

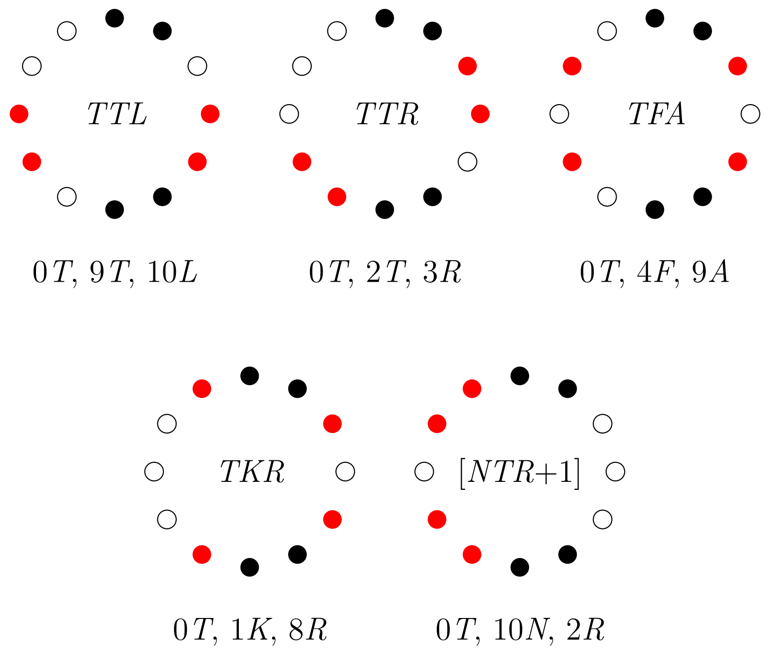

Figure 5: PSTP types with type T tetrachords.

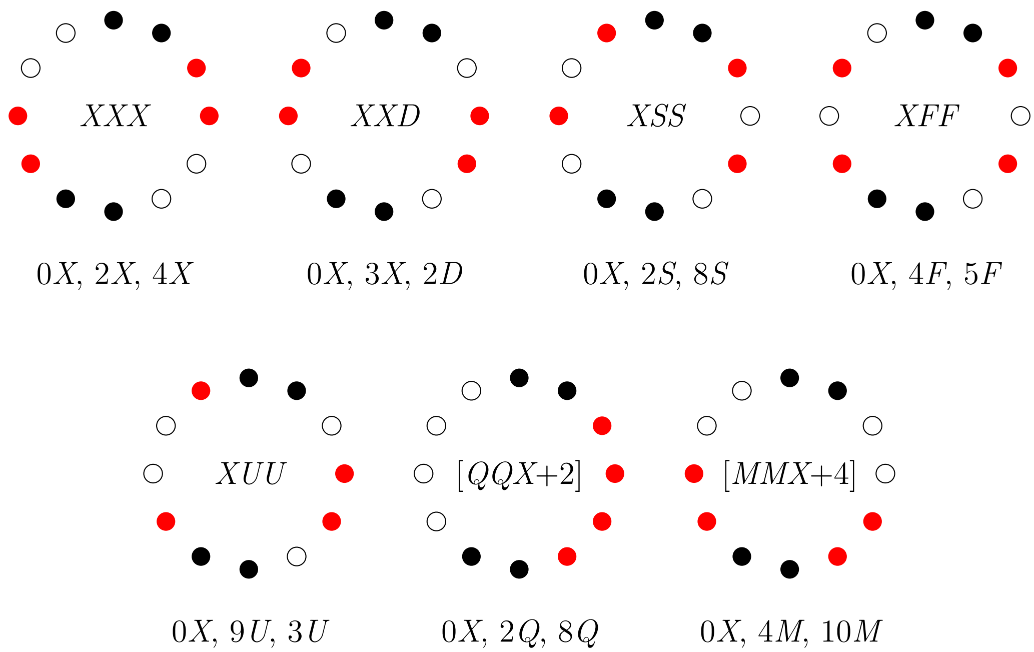

Figure 6: PSTP types with type X tetrachords.

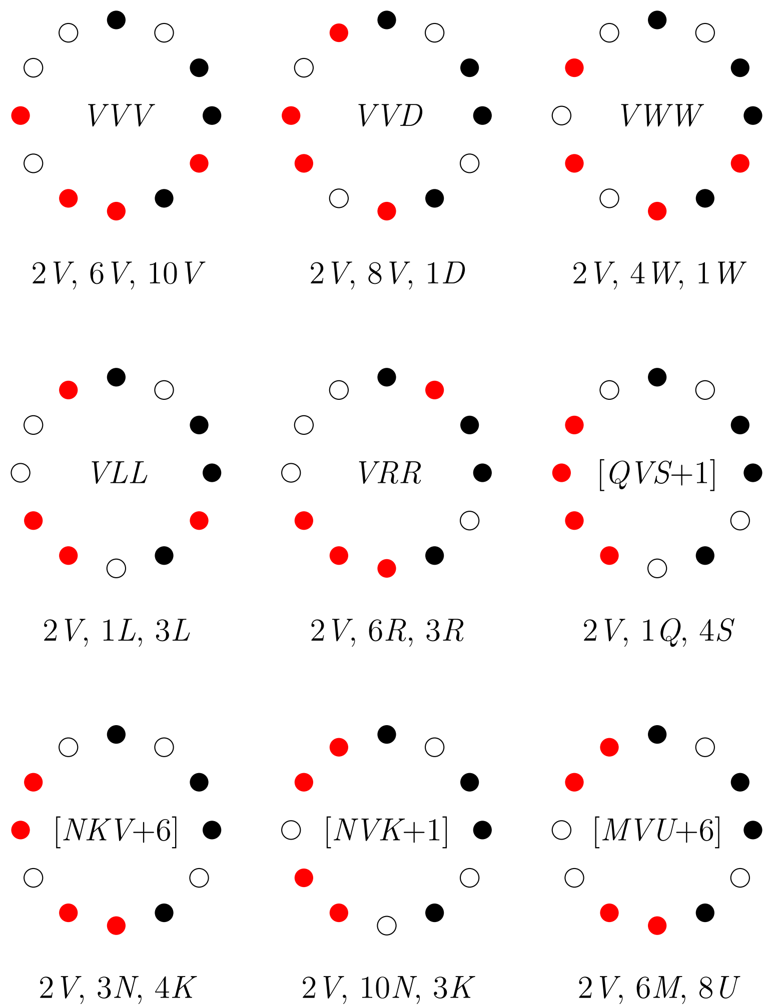

Figure 7: PSTP types with type V tetrachords.

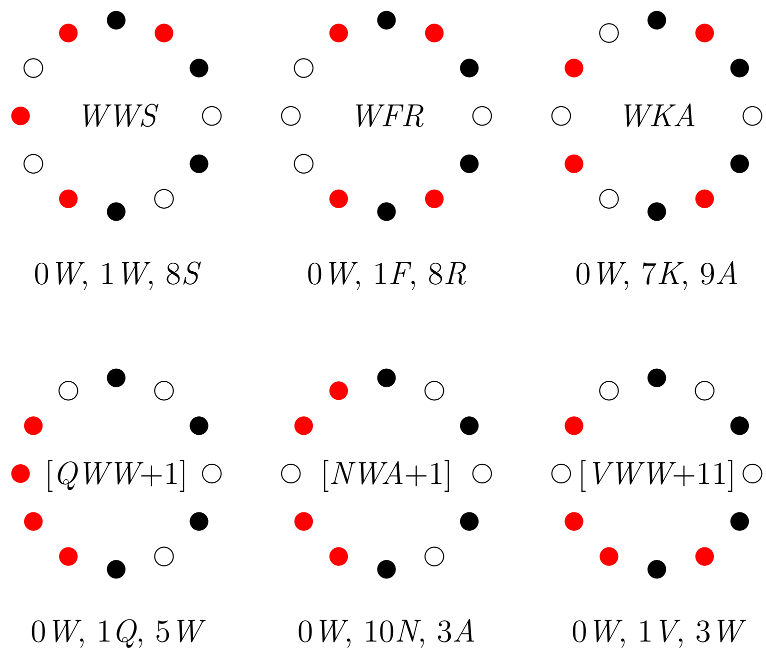

Figure 8: PSTP types with type *W* tetrachords.

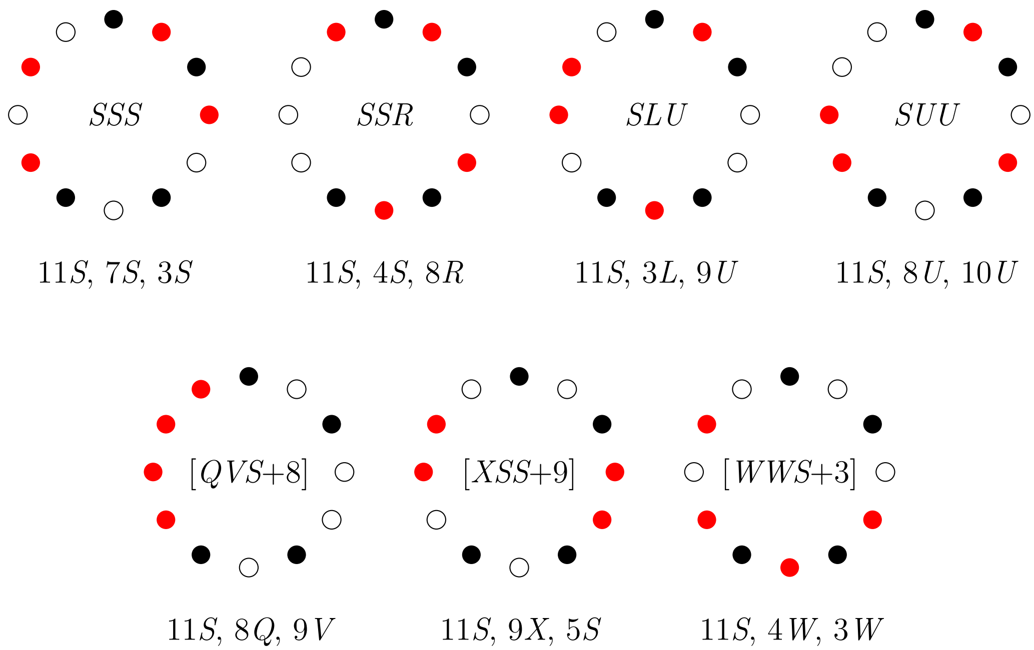

Figure 9: PSTP types with type S tetrachords.

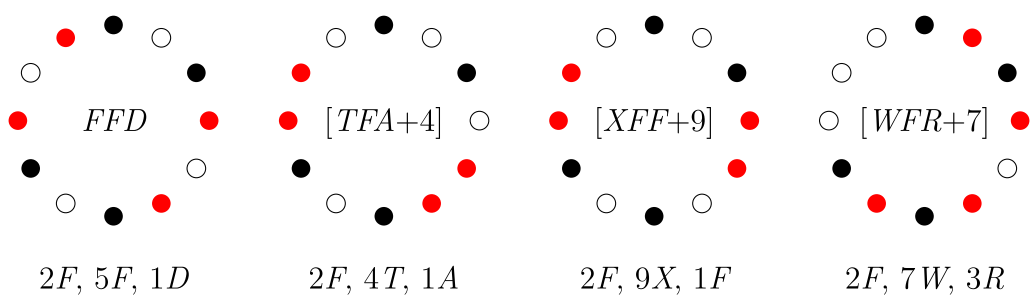

Figure 10: PSTP types with type F tetrachords.

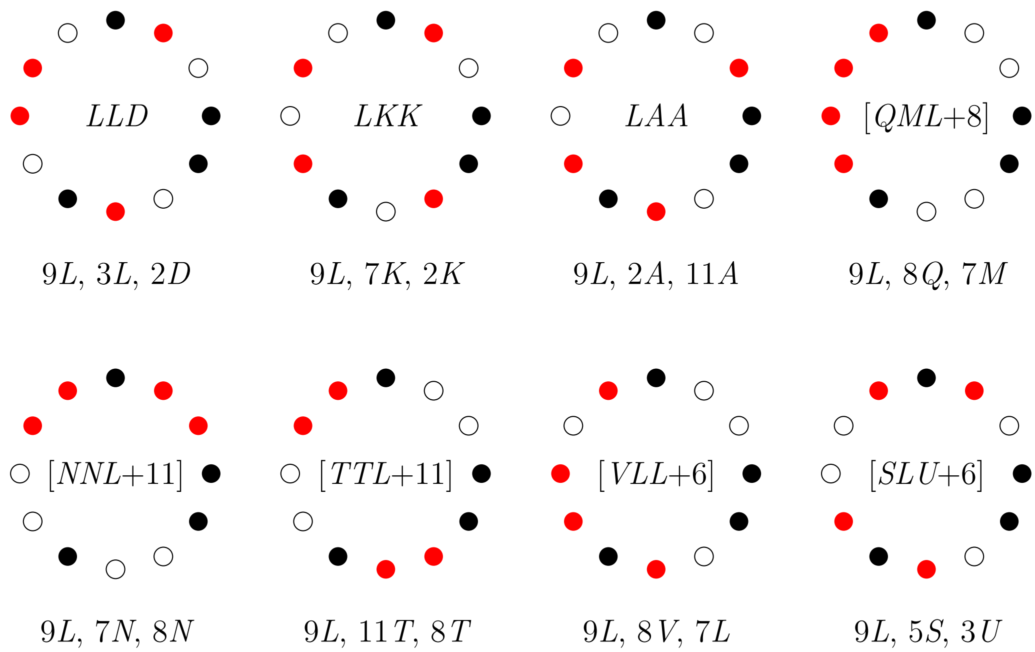

Figure 11: PSTP types with type L tetrachords.

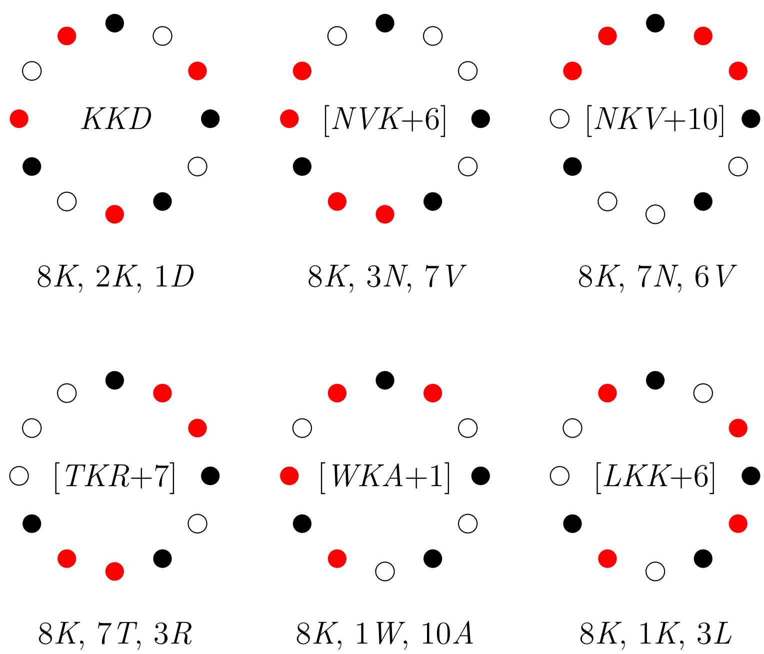

Figure 12: PSTP types with type K tetrachords.

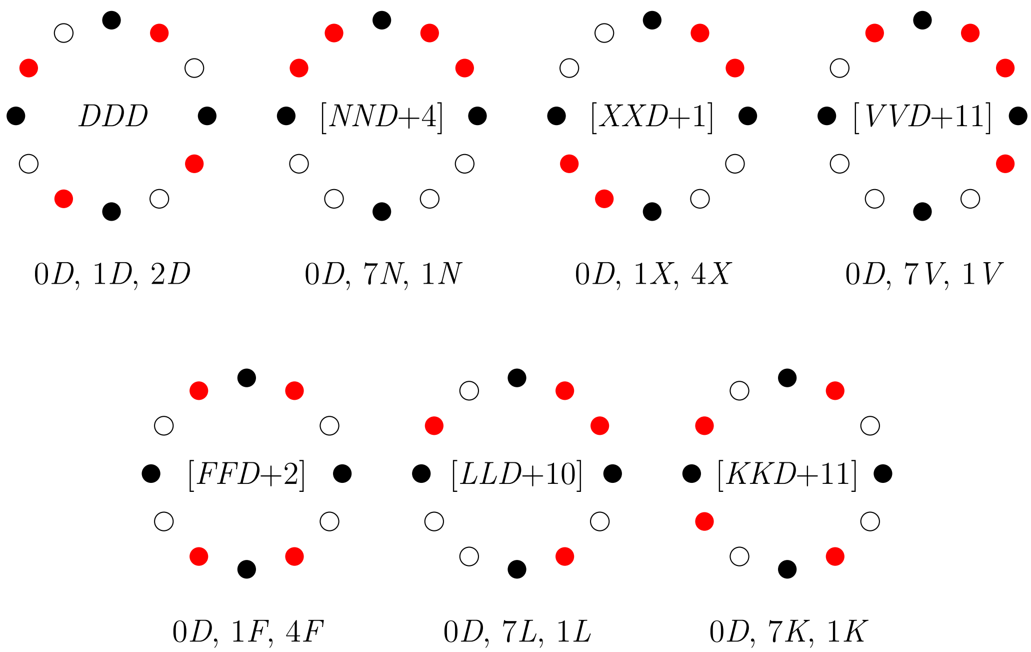

Figure 13: PSTP types with type D tetrachords.

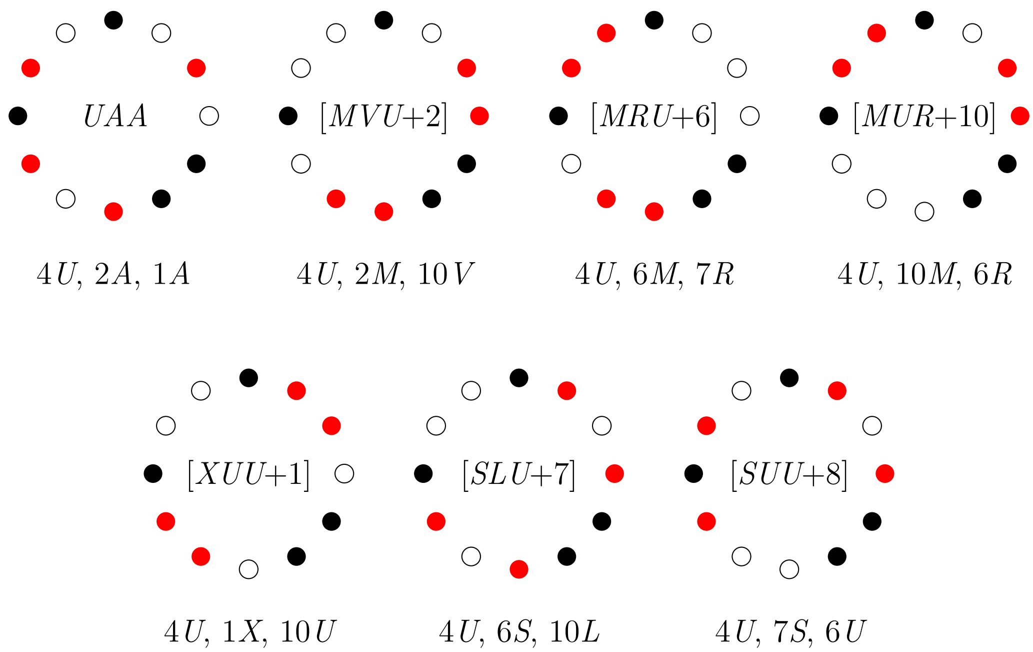

Figure 14: PSTP types with type *U* tetrachords.

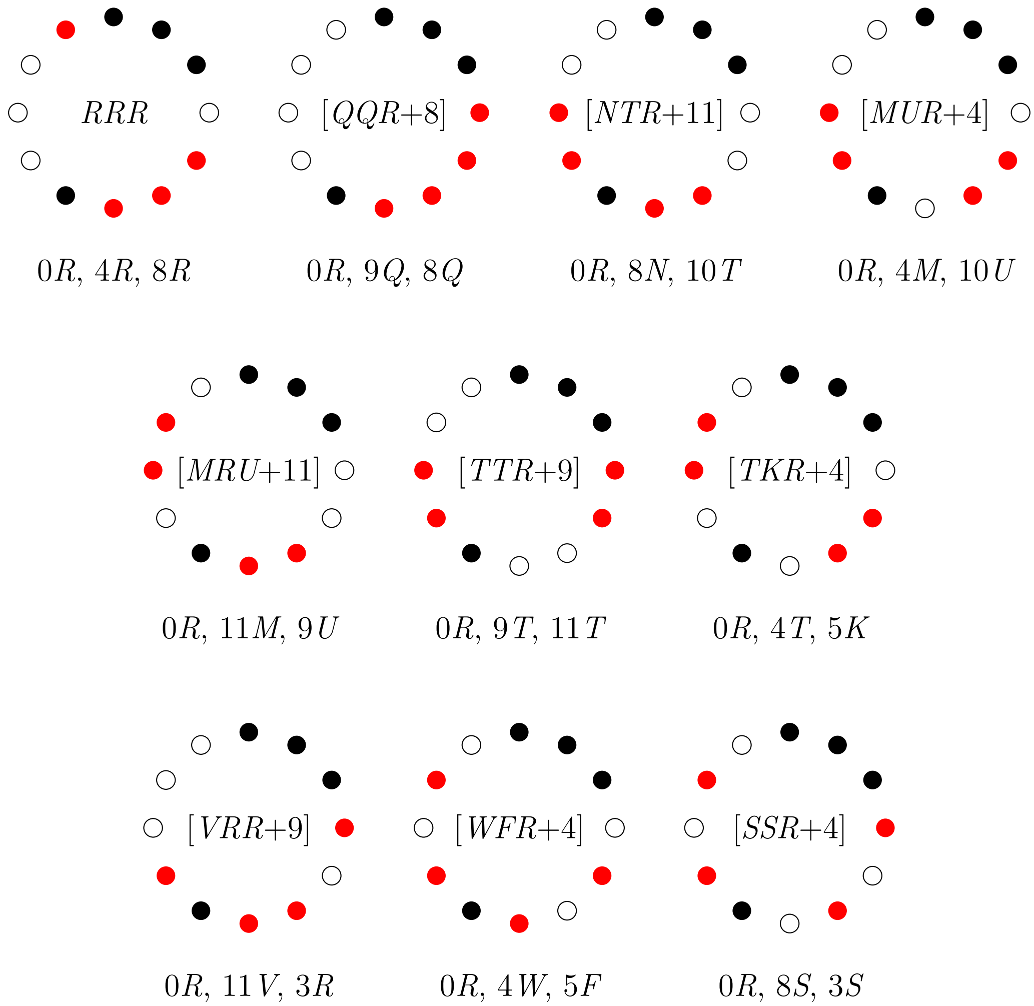

Figure 15: PSTP types with type R tetrachords.

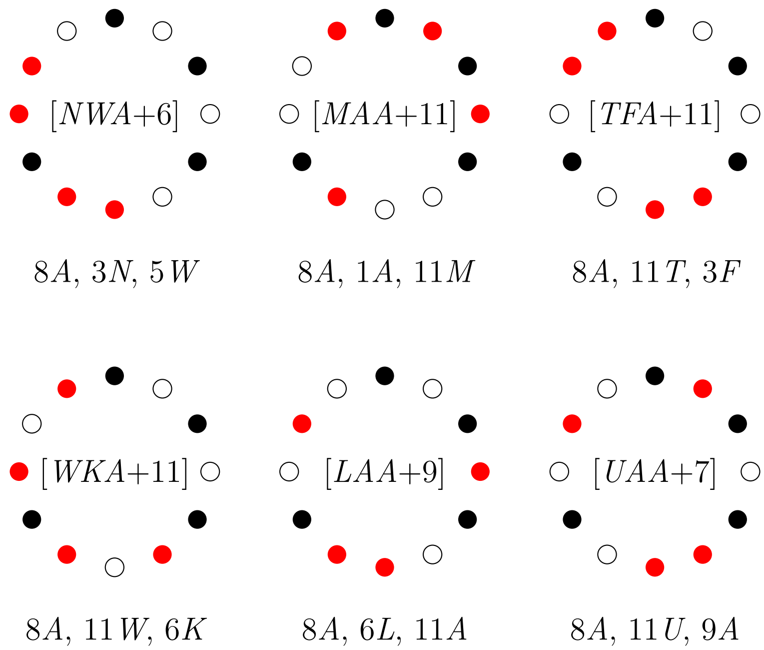

Figure 16: PSTP types with type A tetrachords.

PSTP	*Q*	*N*	*M*	*T*	*X*	*V*	*W*	*S*	*F*	*L*	*K*	*D*	*U*	*R*	*A*
QQQ	0, 4, 8	-	-	-	-	-	-	-	-	-	-	-	-	-	-
QQX	0, 6	-	-	-	4	-	-	-	-	-	-	-	-	-	-
QQR	0, 1	-	-	-	-	-	-	-	-	-	-	-	-	4	-
QMM	0	-	4, 6	-	-	-	-	-	-	-	-	-	-	-	-
QML	0	-	11	-	-	-	-	-	-	1	-	-	-	-	-
QVS	0	-	-	-	-	1	-	3	-	-	-	-	-	-	-
QWW	0	-	-	-	-	-	4, 11	-	-	-	-	-	-	-	-
NNL	-	8, 9	-	-	-	-	-	-	-	10	-	-	-	-	-
NND	-	3, 9	-	-	-	-	-	-	-	-	-	2	-	-	-
NTR	-	9	-	11	-	-	-	-	-	-	-	-	-	1	-
NVK	-	9	-	-	-	1	-	-	-	-	2	-	-	-	-
NKV	-	9	-	-	-	8	-	-	-	-	10	-	-	-	-
NWA	-	9	-	-	-	-	11	-	-	-	-	-	-	-	2
MMX	-	-	0, 6	-	2	-	-	-	-	-	-	-	-	-	-
MVU	-	-	0	-	-	8	-	-	-	-	-	-	2	-	-
MUR	-	-	0	-	-	-	-	-	-	-	-	-	6	8	-
MRU	-	-	0	-	-	-	-	-	-	-	-	-	10	1	-
MAA	-	-	0	-	-	-	-	-	-	-	-	-	-	-	2, 9
TTL	-	-	-	0, 9	-	-	-	-	-	10	-	-	-	-	-
TTR	-	-	-	0, 2	-	-	-	-	-	-	-	-	-	3	-
TFA	-	-	-	0	-	-	-	-	4	-	-	-	-	-	9
TKR	-	-	-	0	-	-	-	-	-	-	1	-	-	8	-
XXX	-	-	-	-	0, 2, 4	-	-	-	-	-	-	-	-	-	-
XXD	-	-	-	-	0, 3	-	-	-	-	-	-	2	-	-	-
XSS	-	-	-	-	0	-	-	2, 8	-	-	-	-	-	-	-
XFF	-	-	-	-	0	-	-	-	4, 5	-	-	-	-	-	-
XUU	-	-	-	-	0	-	-	-	-	-	-	-	3, 9	-	-
	Q	*N*	*M*	*T*	*X*	*V*	*W*	*S*	*F*	*L*	*K*	*D*	*U*	*R*	*A*

Table 2: Pitch-class relations of PSTP constituents.

PSTP	*Q N M T X*	*V*	*W*	*S*	*F*	*L*	*K*	*D*	*U*	*R*	*A*
VVV	- - - - - -	2, 6, 10	-	-	-	-	-	-	-	-	-
VVD	- - - - - -	2, 8	-	-	-	-	-	1	-	-	-
VWW	- - - - - -	2	1, 4	-	-	-	-	-	-	-	-
VLL	- - - - - -	2	-	-	-	1, 3	-	-	-	-	-
VRR	- - - - - -	2	-	-	-	-	-	-	-	3, 6	-
WWS	- - - - - -	-	0, 1	8	-	-	-	-	-	-	-
WFR	- - - - - -	-	0	-	1	-	-	-	-	8	-
WKA	- - - - - -	-	0	-	-	-	7	-	-	-	9
SSS	- - - - - -	-	-	3, 7, 11	-	-	-	-	-	-	-
SSR	- - - - - -	-	-	4, 11	-	-	-	-	-	8	-
SLU	- - - - - -	-	-	11	-	3	-	-	9	-	-
SUU	- - - - - -	-	-	11	-	-	-	-	8, 10	-	-
FFD	- - - - - -	-	-	-	2, 5	-	-	1	-	-	-
LLD	- - - - - -	-	-	-	-	3, 9	-	2	-	-	-
LKK	- - - - - -	-	-	-	-	9	2, 7	-	-	-	-
LAA	- - - - - -	-	-	-	-	9	-	-	-	-	2, 11
KKD	- - - - - -	-	-	-	-	-	2, 8	1	-	-	-
DDD	- - - - - -	-	-	-	-	-	-	0, 1, 2	-	-	-
UAA	- - - - - -	-	-	-	-	-	-	-	4	-	1, 2
RRR	- - - - - -	-	-	-	-	-	-	-	-	0, 4, 8	-
	Q N M T X	*V*	*W*	*S*	*F*	*L*	*K*	*D*	*U*	*R*	*A*

Table 3: Pitch-class relations of PSTP constituents (continued).

4 Musical Significance

In my opinion, the primary significance of any style, technique, or system of musical composition subsists in the works of its practitioners. John L. Baker of Vancouver, Canada is a composer who has produced several works using this technique, including those listed in Table 4. This website includes a performance and score of his 2009 Prelude, Toccata, and Fugue.

title	date	instrumentation	duration
Passacaglia and Fugue	1993	sop. recorder, tenor crumhorn	4m
Prelude in E	1994	piano solo	3m 30s
Sarabanda *from* Homage in B	1996	piano solo	3m 10s
Nocturne with Balletto	1997	treble, tenor, and bass viols	6m
Fantasy No. 1	2002	piano solo	5m 45s
Fantasy No. 2	2003	wind quintet (fl, ob, cl, hn, bn)	11m
Fantasy No. 3	2006	ob, cl, tbn, vln, cello	6m 30s
Six Inventions	2007	flute, viola, and piano	8-9m
The Cantor-Schröder-Bernstein Theorem, a cantata	2008	12 voices unaccompanied	8m
Prelude, Toccata, and Fugue	2009	piano solo	4m

Table 4: PSTP compositions by John L. Baker.

Here are a few desultory observations on the properties of PS tetrachords and PSTPs.

- In case the constituent tetrachords of a PSTP share an axis of symmetry (as, for example, those of type *QML*), their mutual symmetry can structure a harmonic progression (at or below the surface), thus:

- Say that a 12-tone row x *realizes* a PSTP y if the constituent tetrachord types of y comprise the first, middle, and last four elements of x. Most PSTPs[9] can

[9] All except those of types *TTR, DDD,* and *RRR.* See appendix A.

be realized by a fully combinatorial 12-tone row.[10] For example, the the fully combinatorial row

$$\langle 0, 2, 7, 9, 4, 5, 1, 8, 10, 6, 3, 11 \rangle$$

realizes the PSTP

$$\{\{0, 7, 2, 9\}, \{11, 6, 3, 10\}, \{4, 1, 8, 5\}\} = \{0Q, 11M, 1L\}.$$

The PSTPs admitting such realizations are enumerated in Appendix A.

My interest in PS tetrachords was triggered by Elliott Antokoletz's[11] detailed analysis of Bartók's use of pitch-symmetry in general and of the *S, W,* and X[12] type tetrachords in particular. However, Bartók does not use these PS tetrachords as part of a PSTP-like system or technique: in fact, the way in which he combines S and W types is incompatible with completion by a third PS tetrachord to form a PSTP; likewise for his combinations of W and X types, which cannot occur together at all in a PSTP. (See Table 2.) Hence, for what it may be worth, I consider that the PSTP technique described here, although informed by the growing use of inversionally symmetrical pitch formations in European music from the latter part of the nineteenth century up to Bartók's time,[13] is nevertheless original with me.

[10]For definitions relating to combinatoriality, see "Pitch-Symmetric Tetrachords, etc" on this website, p 9.

[11] *The Music of Béla Bartók* (University of California Press, 1984).

[12]Following George Perle ("Symmetrical Formations in the String Quartets of Béla Bartók" [*Music Review* no. 16 (November 1955), pp. 300-312]) and Leo Treitler ("Harmonic Procedure in the *Fourth Quartet* of Béla Bartók" [*Journal of Music Theory* v.3, no.2 (November 1959), pp. 292-297]), Antokoletz refers to PS tetrachords of types *S, W,* and *X* as cells X, Y, and Z respectively.

[13]Antokoletz, pp.4-25.

A PSTPs Realizable by Fully Combinatorial Rows

It is well known that the fully combinatorial 12-tone rows are precisely those whose first and second halves are both hexachords of one of the types P1, P2, P3, P4, P5, or P6 in George Perle's enumeration,[14] as shown in Figure 17. For a PTSP to be

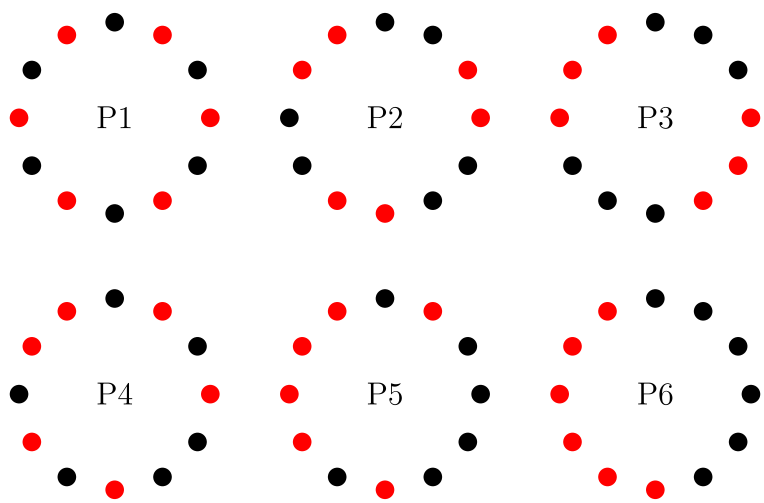

Figure 17: Hexachords (Circle-of-Fifths Order) for Fully Combinatorial Rows.

realizable by a fully combinatorial row in the sense given above (p 22), it is merely necessary for two of its constituent tetrachords to be included individually in the two hexachords of one of these types. One can determine whether this is so for any particular PSTP by inspection, comparing Figures 2-16 with Figure 17. Consider for example the PSTP QML as shown in Figure 2. There are three orientations of the diagram for P6 in Figure 17 in which its black hexachord covers the positions of the black tetrachord in Figure 2, namely counter-clockwise rotations by 0, 1, or 2 positions. Rotated counter-clockwise by one position, the red hexachord of Figure 17 covers the positions of the M (red) tetrachord of QML in Figure 2; this shows that there are fully combinatorial tone rows beginning with the elements of $0Q$ and ending with the elements of $11M$ (as in the example on p 23). On the other hand, rotated counter-clockwise by 0 or 2 positions, the red hexachord of Figure 17 does not cover

[14]Perle's enumeration is shown in full in "Hauer's Tropes and the Enumeration of Twelve-Tone Hexachords" on this website. Note that the diagrams there are in circle-of-semitones order, so that the appearances of the diagrams for P4 and P6 are interchanged.

the position of either the M (red) or the L (white) tetrachord of QML in Figure 2; this shows that $0QML$ is not realizable by any other fully combinatorial rows based on P6 beginning with the elements of $0Q$.

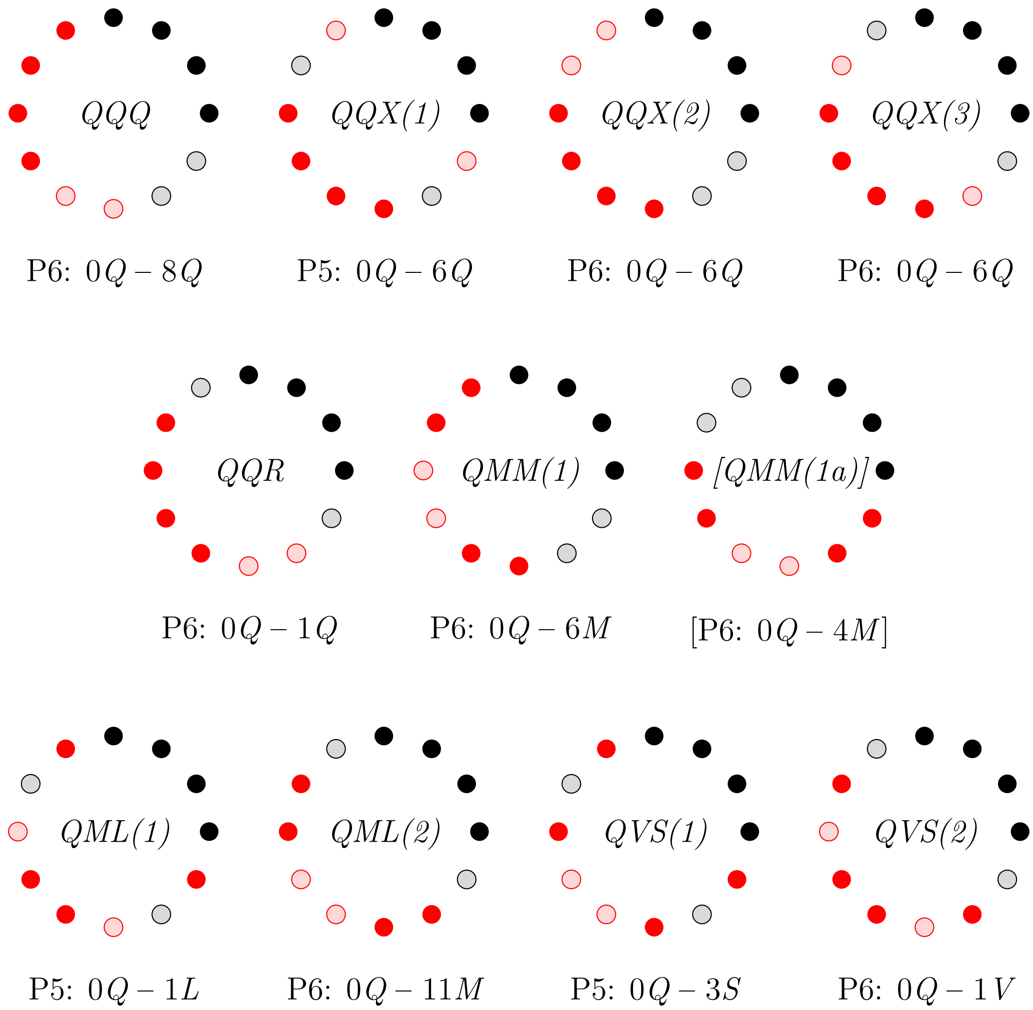

Figure 18: PTSPs realizable by rows with type Q incipit.

Figures 18-31 exhibit the results of all such comparisons, and therefore show all the ways in which PSTPs can be realized by fully combinatorial tone rows. Notation for these figures is generally as explained on p 3, with the following alterations.

The black dots in each circle, which are in the same positions as in the corresponding diagram in Figures 2-16, indicate the initial tetrachord ("incipit", let us say) of the (directly) realizing tone rows. The red dots, which correspond either to the red or to the white dots in the corresponding diagram in Figures 2-16, indicate the final tetrachord of the same tone rows. The other four dots, coloured grey or pink, combine with the black or red dots respectively to make up the initial and final hexachords of those rows. Internal sub-captions (like "P6: $0Q - 11M$" in Figure 18) give the hexachord type and name the initial and final tetrachords. Note that, if a PSTP in this listing has initial and final tetrachords of different types, then it will appear in the figure for each type as incipit.

Note also that there are no realizations with type D incipit.

The assertion above that "Figures 18-31 exhibit ... all the ways in which PSTPs can be realized by fully combinatorial tone rows" requires some clarification. The realizing tone rows directly represented in those figures, for example

$$\langle 0, 7, 2, 9, 4, 11, 8, 3, 6, 1, 10, 5\rangle, \tag{A.1}$$

which realizes the PSTP $0QMM$ as shown in the diagram *QMM(1)* of Figure 18, is merely one member of the complex of 48 rows comprising all those obtainable from it by combinations of transposition, inversion, and reversal, and the diagram in question should be considered to express the fact that all the member rows of all the complexes including rows thus directly represented realize some PTSP of type *QMM.* It follows that each diagram indicates the realizability of a PTSP type in a particular way by up to $4! \cdot 2 \cdot 2 \cdot 4! \cdot 48 = 110,592$ fully combinatorial 12-tone rows.[15]

As a consequence of this interpretation of the diagrams, some possible coverings of tetrachords in Figures 2-16 by the hexachords of Figure 17 are redundant. For example, one might suppose that, in addition to the covering of $0QMM$ (Figure 2) by P6 indicated by *QMM(1)* (Figure 18), another covering *[QMM(1a)]* (Figure 18), obtained by rotating P6 two positions counter-clockwise, would also be relevant. However, because *[QMM(1a)]* is a reflection of *QMM(1),* the set of fully combinatorial rows realizing these two PTSPs is in fact identical. (A.1), for example is the I_0 form of

$$\mathrm{P}_0 = \langle 0, 5, 10, 3, 8, 1, 4, 9, 6, 11, 2, 7\rangle,$$

the P_9 form of which ($= \langle 9, 2, 7, 0, 5, 10, 1, 6, 3, 8, 11, 4\rangle$) realizes *[QMM(1a)].*

[15] The factor $4! \cdot 2 \cdot 2 \cdot 4! = 2,304$ corresponds to the (independent) permutations of $\{0, 7, 2, 9\}, \{4, 11\}, \{5, 8\},$ and $\{6, 1, 10, 5\}$—the sets represented by the four colours of dots.

Apart from *[QMM(1a)]*, included just to illustrate the foregoing point, there are two exceptions to the avoidance of this sort of redundancy in Figures 18-31:

- As mentioned above, if a PSTP in this listing has initial and final tetrachords of different types, then it will appear in the figure for each type as incipit. In this case the second appearance are marked by "*[ ]*" around the name in the centre of the circle.
- Because *NVK* and *NKV* (Figure 3) are related by inversion as PSTP types, it follows that corresponding digrams in Figure 19, etc indicate the same sets of realizing tone rows. Likewise for *MUR* and *MRU* (Figure 4). In this case all appearances of *NKV* and *MRU* are marked by (possibly additional) "*[ ]*" around the name in the centre of the circle.

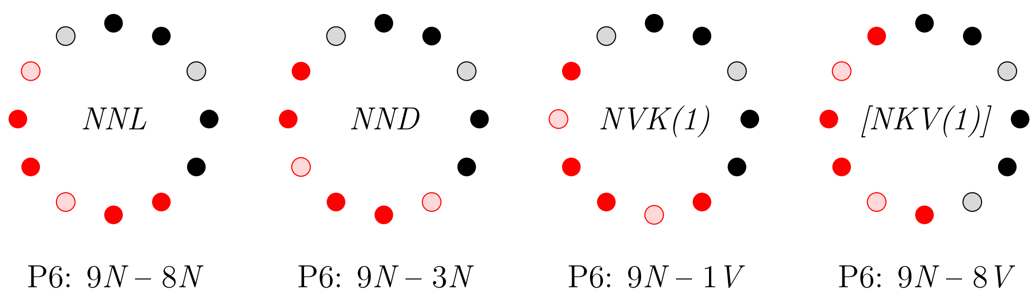

Figure 19: PTSPs realizable by rows with type N incipit.

To summarize, this appendix answers three types of question:

- Which PSTPs are realizable by fully combinatorial 12-tone rows? *Answer: All except those of types TTR, DDD, or RRR.*
- In what ways can a PSTP be realized by a fully combinatorial 12-tone row? That is, to which of the patterns of Figure 17 must a realizing row conform, and how can the constituent tetrachords of the PSTP be distributed within the row?
- Given a fully combinatorial row comprising three pitch-symmetric tetrachords in sequence, which PSTP does it realize? Note that this question always has a unique answer except for the alternation between *NVK* and *NKV* (respectively *MUR* and *MRU*).

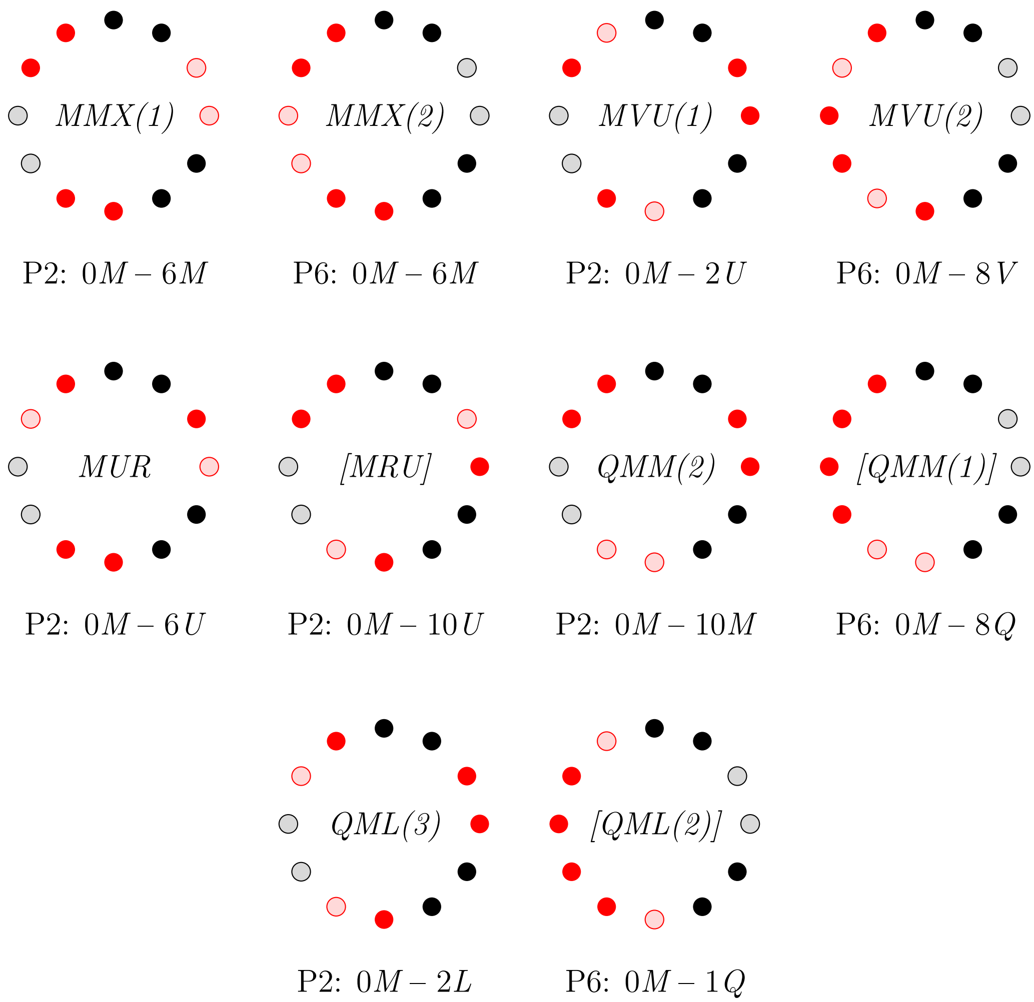

Figure 20: PTSPs realizable by rows with type M incipit.

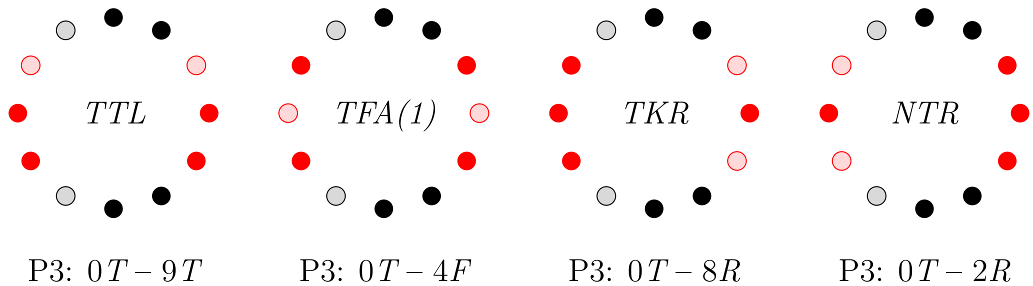

Figure 21: PTSPs realizable by rows with type T incipit.

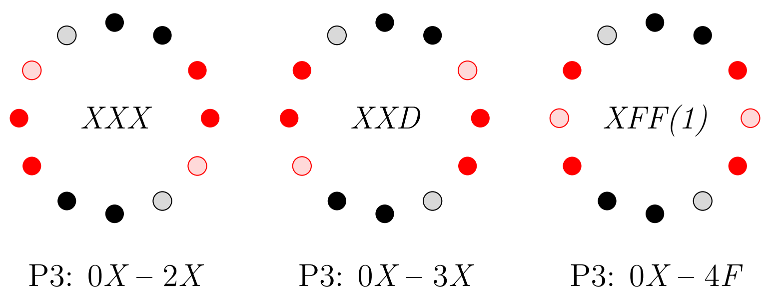

Figure 22: PTSPs realizable by rows with type X incipit.

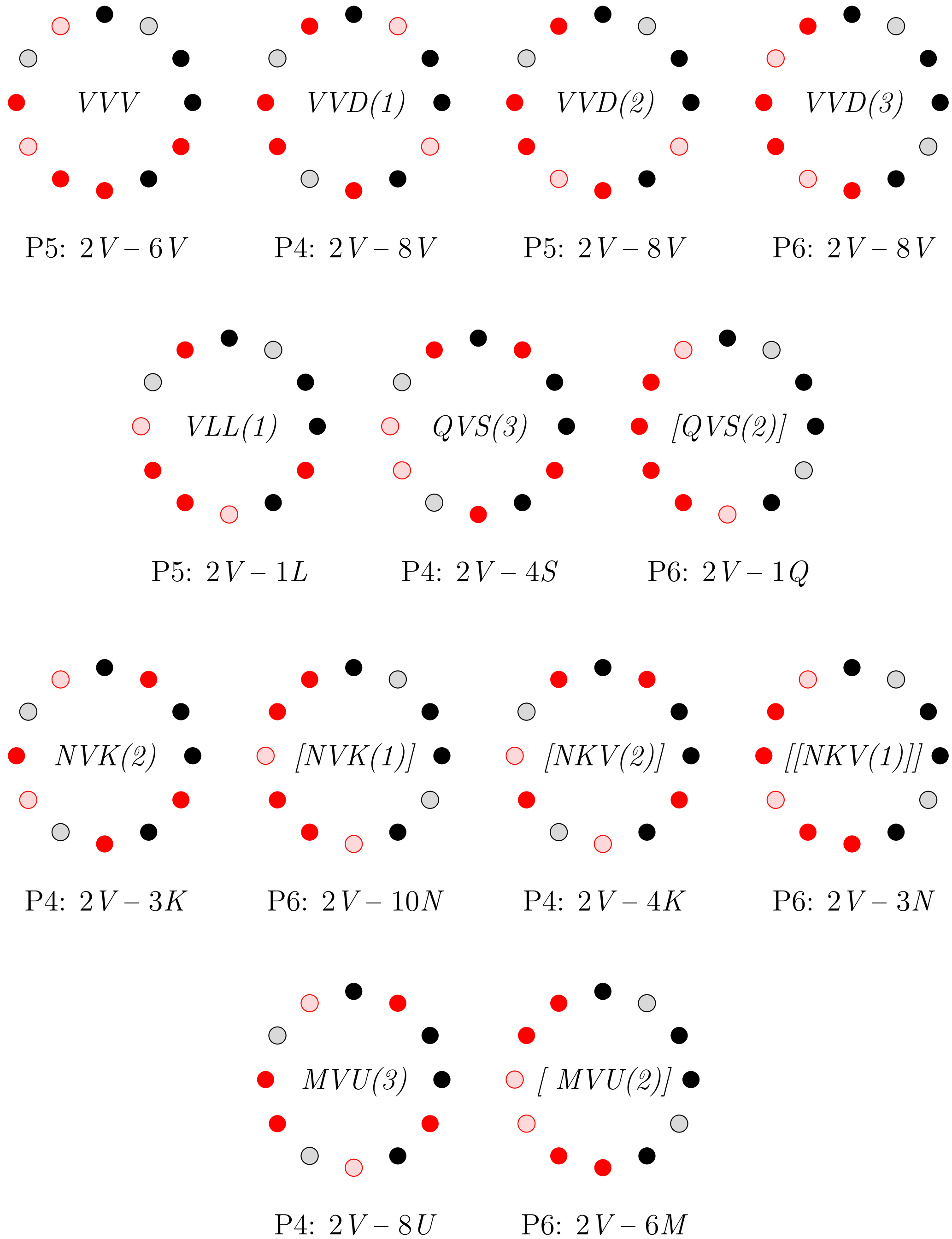

Figure 23: PTSPs realizable by rows with type V incipit.

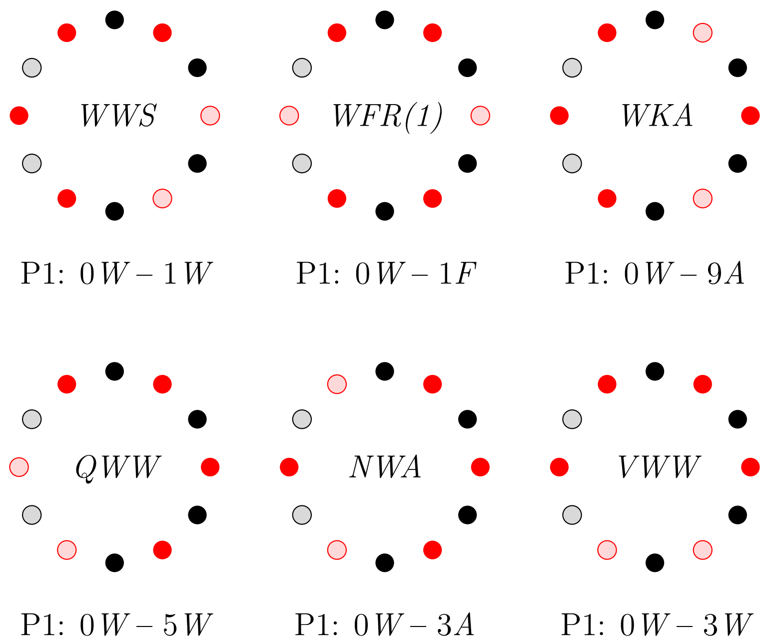

Figure 24: PTSPs realizable by rows with type W incipit.

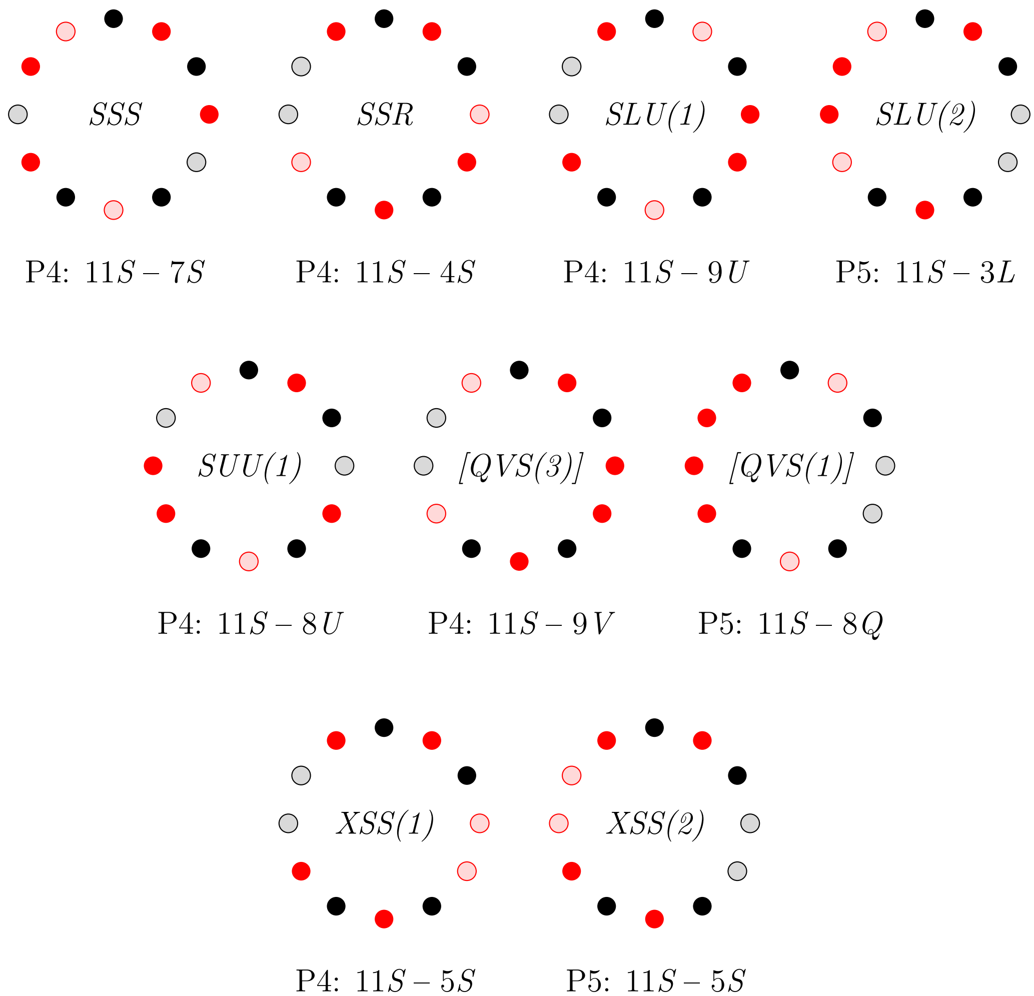

Figure 25: PTSPs realizable by rows with type S incipit.

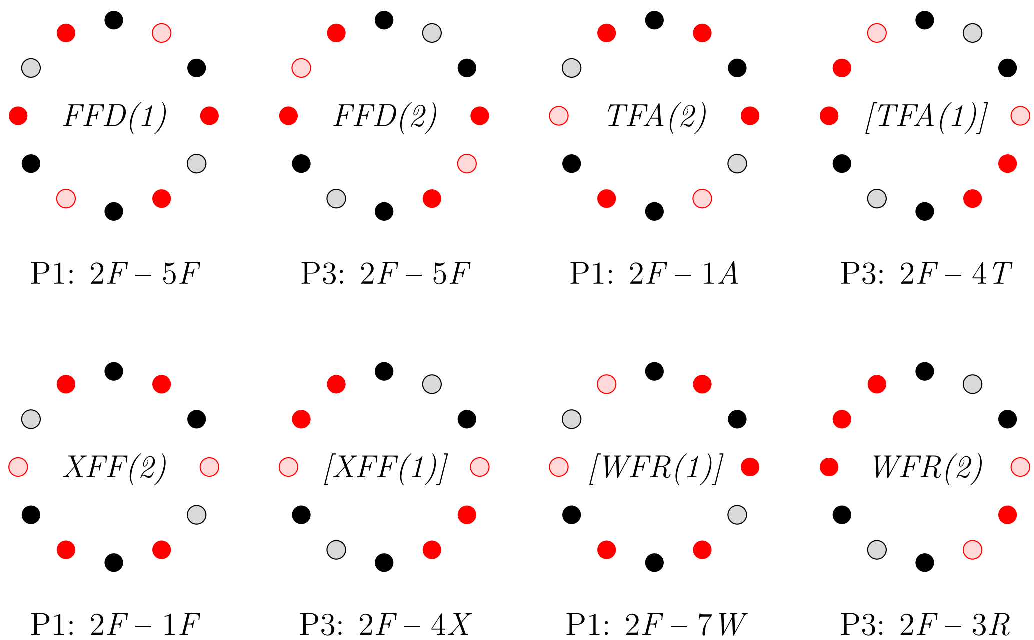

Figure 26: PTSPs realizable by rows with type F incipit.

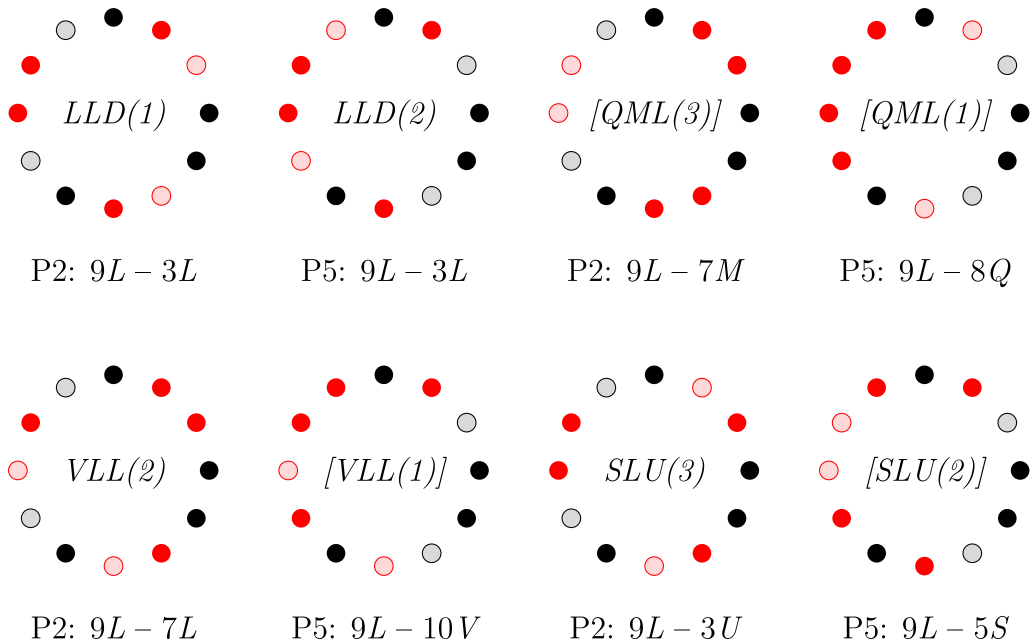

Figure 27: PTSPs realizable by rows with type L incipit.

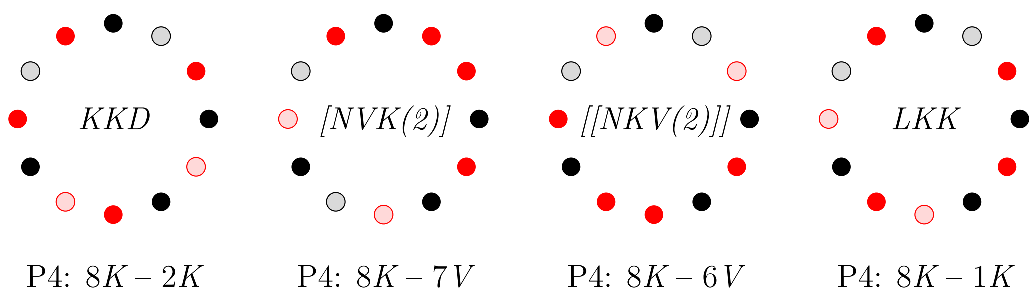

Figure 28: PTSPs realizable by rows with type K incipit.

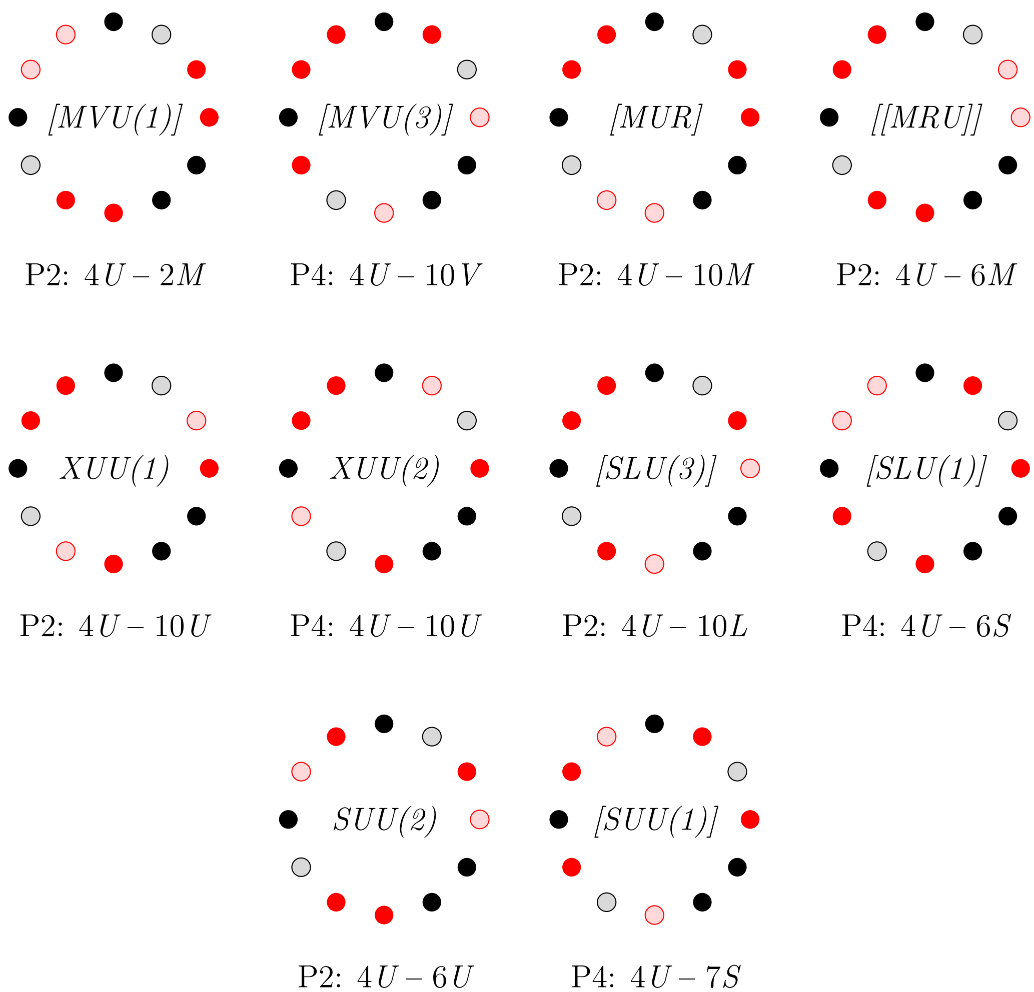

Figure 29: PTSPs realizable by rows with type U incipit.

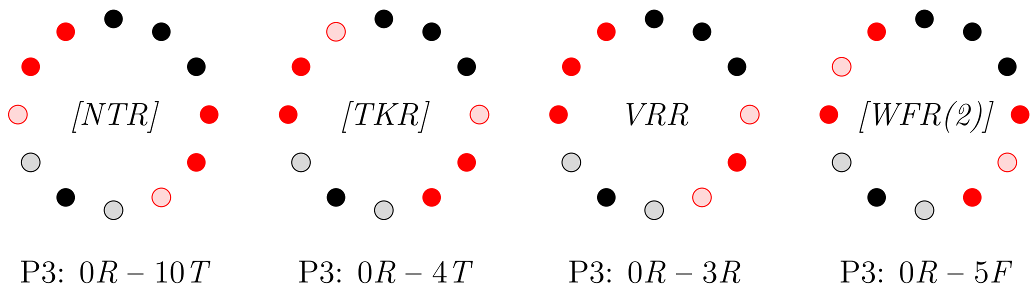

Figure 30: PTSPs realizable by rows with type R incipit.

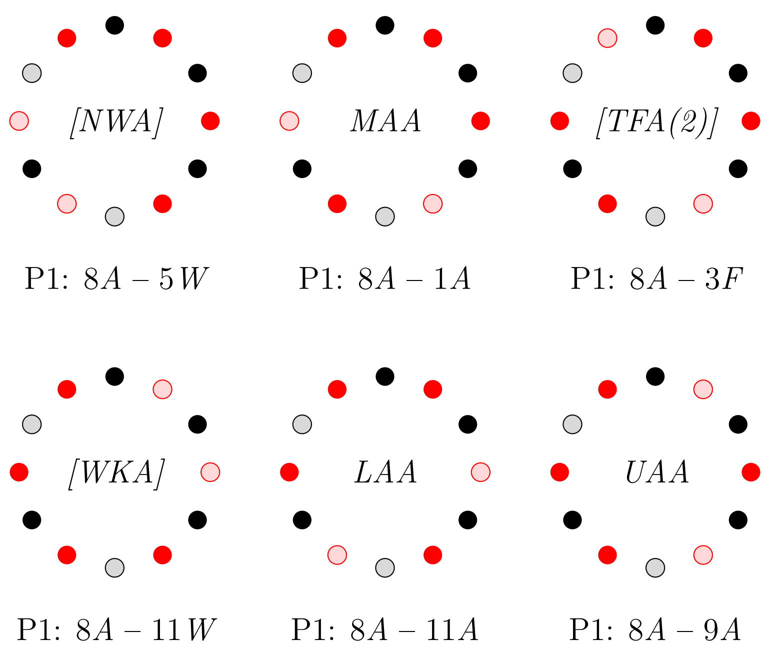

Figure 31: PTSPs realizable by rows with type A incipit.

B Enumerating the PSTPs

The Pascal[16] program below produces something closely analogous to the listing in Tables 2 and 3.[17] It first generates a linked list of (PS) `Tetrachord` records in the order *Q, N, . . . , A*; then with each in turn in the 0 position,[18] tries to fit the same tetrachord (then each of the following ones) into the remaining positions. When it finds a fit for two tetrachords, it then checks if the remaining four positions are also PS (i.e. on the list); if so, it reports.

Notice that the `Tetrachord` record contains all that is needed to generate and report and also (in the member array `content`) the sets of positions occupied when the tetrachord is based at 0, 1, . . . , 11 respectively. This makes for quick testing: `if T1^.content[i1] <= U1` (`<=` means 'is a subset of'); `if T2^.content[i2] = U2`.

```
program Symm4chdPrtns;

   const
      All = [0..11];

   type
      PC = 0..11;
      PCset = set of PC;
      interval = 0..5;
      link = ^Tetrachord;
      Tetrachord = record
            root: PC;
            name: string[1];
            xlimit: PC;
            content: array[PC] of PCset;
            next: link
         end;
```

[16] Jensen, Kathleen and Niklaus Wirth, *Pascal User Manual and Report,* 2nd ed.: Springer-Verlag, 1974 (corr. 1978).

[17] The program's output actually comprises 53 lines, 6 of which are (shifted) duplicates which have to be eliminated by hand—e.g. "`0Q 11Q 3R`" duplicating "`0Q 1Q 4R`".

[18] In circle-of-fifths order—likewise for all references to position in this appendix.

```
procedure NewSymmTchord_A (R: PC; N: string; X: PC;
                                first, second: interval; var where: link);
      {On return, where points to the new record, the .next field of which
         points where where used to.}
   var
      i0, i1, i2, i3: PC;
      oldwhere: link;
   begin
      oldwhere := where;
      new(where);
      with where^ do
         begin
            next := oldwhere;
            root := R;
            name := N;
            xlimit := X;
            for i0 := 0 to X do
                  {X is limit due to internal symmetry}
               begin
                  i1 := (i0 + first) mod 12;
                  i2 := (i1 + second) mod 12;
                  i3 := (i2 + first) mod 12;
                  content[i0] := [i0, i1, i2, i3]
                               {symmetric 4-chord transposed by i0}
               end
         end
   end;
```

```
procedure NewSymmTchord_B (R: PC; N: string; X: PC;
                                first: interval; var where: link);
      {On return, where points to the new record, the .next field of which
         points where where used to.}
   var
      second, i0, i1, i2, i3: PC;
      oldwhere: link;
   begin
      second := 6 - first;
      oldwhere := where;
      new(where);
      with where^ do
         begin
            next := oldwhere;
            root := R;
            name := N;
            xlimit := X;
            for i0 := 0 to X do
                  {X is limit due to internal symmetry}
               begin
                  i1 := (i0 + first) mod 12;
                  i2 := (i1 + first) mod 12;
                  i3 := (i2 + second) mod 12;
                  content[i0] := [i0, i1, i2, i3]
                                 {symmetric 4-chord transposed by i0}
               end
         end
   end;
```

```
procedure writePCset (X: PCset);
   var
      i: PC;
      first: Boolean;
   begin
      first := true;
      write('[');
      for i := 0 to 11 do
         if i in X then
             if first then
                begin
                   first := false;
                   write(i : 1)
                end
             else
                write(i : 3);
      write('] ');
   end;

procedure ReportPartition (S0, S1, S2: link; j0, j1, j2: PC);
   begin
      with S0^ do
         write((root + 7*j0) mod 12 : 2, name, ' ');
      with S1^ do
         write((root + 7*j1) mod 12 : 2, name, ' ');
      with S2^ do
         write((root + 7*j2) mod 12 : 2, name, ' ');
      write(' :  ');
      writePCSet(S0^.content[j0]);
      writePCSet(S1^.content[j1]);
      writePCSet(S2^.content[j2]);
      writeln;
   end;

var
   i1, i2, starti2: PC;
   U1, U2: PCset;
   Tlist, T0, T1, T2: link;
```

```
begin

{Build list of pitch-symmetric tetrachords,
          in reverse of output canonical order: }
   Tlist := nil;
      {sets in circle-of-fifths order <-> circ-of-st order}
   NewSymmTchord_B( 8, 'A', 11, 2, Tlist);
      {Aug+Tt : [0, 2, 4, 8] <-> [8, 0, 2, 4] }
   NewSymmTchord_B( 0, 'R', 11, 1, Tlist);
      {Quintal+m9(Tt) : [0, 1, 2, 7] <-> [0, 7, 2, 1] }
   NewSymmTchord_A( 4, 'U', 11, 4, 1, Tlist);
      {4 PM7+m6 : [0, 4, 5, 9] <-> [4, 11, 0, 3] }
   NewSymmTchord_A( 0, 'D',  2, 3, 3, Tlist);
      {0 dd7 : [0, 3, 6, 9] <-> [0, 3, 6, 9]; 4-fold symmetry}
   NewSymmTchord_A( 8, 'K', 11, 3, 2, Tlist);
      {mixed cluster : [0, 3, 5, 8] <-> [8, 9, 11, 0] }
   NewSymmTchord_A( 9, 'L', 11, 3, 1, Tlist);
      {9 M+m3 : [0, 3, 4, 7] <-> [9, 0, 1, 4] }
   NewSymmTchord_A( 2, 'F',  5, 2, 4, Tlist);
      {Fr x6 : [0, 2, 6, 8] <-> [2, 6, 8, 0]; 2-fold symmetry}
   NewSymmTchord_A(11, 'S', 11, 2, 3, Tlist);
      {semitone cluster : [0, 2, 5, 7] <-> [11, 0, 1, 2] }
   NewSymmTchord_A( 0, 'W', 11, 2, 2, Tlist);
      {whole-tone cluster : [0, 2, 4, 6] <-> [0, 2, 4, 6] }
   NewSymmTchord_A( 2, 'V', 11, 2, 1, Tlist);
      {2 Pm7+M6 : [0, 2, 3, 5] <-> [2, 9, 11, 0] }
   NewSymmTchord_A( 0, 'X',  5, 1, 5, Tlist);
      {crossed tritones : [0, 1, 6, 7] <-> [0, 1, 6, 7]; 2-fold symm}
   NewSymmTchord_A( 0, 'T', 11, 1, 4, Tlist);
      {0 PM7+Tt : [0, 1, 5, 6] <-> [0, 6, 7, 11] }
   NewSymmTchord_A( 0, 'M', 11, 1, 3, Tlist);
      {0 MM7 : [0, 1, 4, 5] <-> [0, 4, 7, 11] }
   NewSymmTchord_A( 9, 'N', 11, 1, 2, Tlist);
      {9 mm7 : [0, 1, 3, 4] <-> [9, 0, 4, 7] }
   NewSymmTchord_A( 0, 'Q', 11, 1, 1, Tlist);
      {Quintal-Quartal set : [0, 1, 2, 3] <-> [0, 7, 2, 9] }
```

```
{Report partitions, searching in direct canonical order:}
   T0 := Tlist;
   while T0 <> nil do
      begin
         U1 := All - T0^.content[0];
         T1 := T0;
         while T1 <> nil do
            begin
               for i1 := 1 to T1^.xlimit do
                  if T1^.content[i1] <= U1 then
                     begin
                        U2 := U1 - T1^.content[i1];
                        starti2 := i1;
                           {starti2 := i1 + 1 is more the right idea,
                              but wrong if
                                 i1 = T1^.xlimit;
                              to deal with this, it's easier just to let
                              the test at (*) below fail the first time
                              (as clearly it must), and then go on with
                                 i2 = i1 + 1 or
                                 T2 = T1^.next,
                              whichever is appropriate.}
                        T2 := T1;
                        while T2 <> nil do
                           begin
                              for i2 := starti2 to T2^.xlimit do
                  {(*)}          if T2^.content[i2] = U2 then
                                     ReportPartition(T0, T1, T2, 0, i1, i2);
                                 starti2 := 1;
                                 T2 := T2^.next
                           end;
                     end;
               T1 := T1^.next
            end;
         T0 := T0^.next
      end

end.
```