Pitch-Symmetric Tetrachords, etc

Polytrope

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Contents

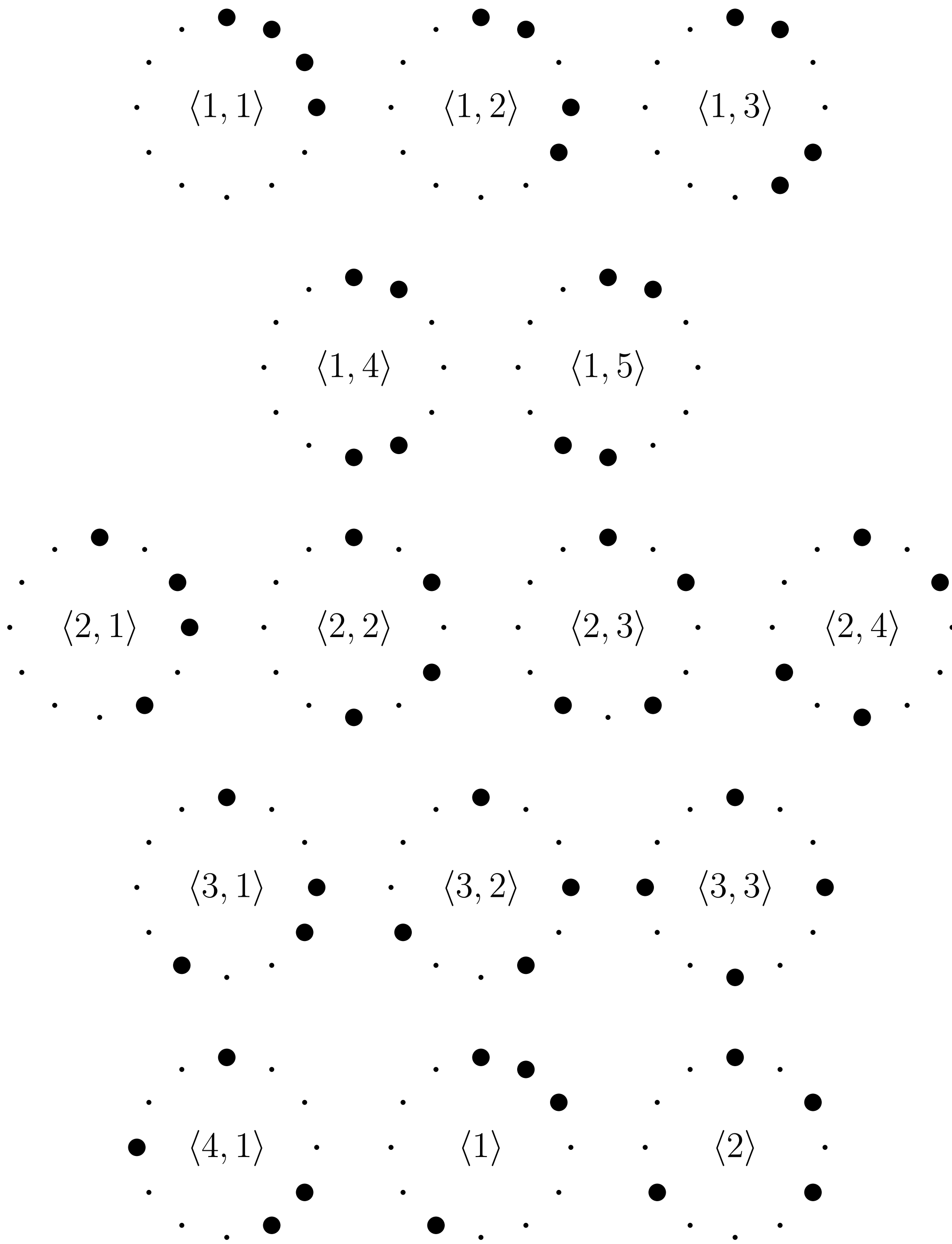


Figure 1: PS 4-chord types, $n = 12$.

1 Pitch-Symmetric Tetrachords in a Twelve-Tone Scale[1]

A particular technique of twelve-tone musical composition which I frequently employ makes extensive use of four-element pitch-class sets[2]—*tetrachords* or *4-chords*—which are *inversionally pitch-symmetric (PS)*. These are sets like

{A, C, E, G}, in which A is nominally as far below C (three semitones) as G is nominally above E; or

{C, D, E, G♯}, in which C is nominally as far below D (a whole tone) as E is nominally above it, and G♯ is a tritone away from D, so equally distanced from C and E.

As for twelve-tone composition in general, the full set of available pitch classes comprises the conventional twelve.[3]

Figure 1 shows diagrammatically all types of these PS tetrachords in the twelve-pitch-class context; in each circle of Figure 1, the twelve positions represent distinct pitch classes, the four marked with heavier dots representing those constituting a particular PS tetrachord. In the centre of each circle is a symbol specifying the *type* of the tetrachord represented. Each of these type specifications is either of the form

$\langle u, v\rangle$, specifying u steps clockwise from the first (12 o'clock) tetrachord element to the second,
v steps from the second to the third,
and again u steps from the third to the fourth,
(so $12 - (2u + v)$ steps from the fourth on again to the first);
(1.1)

[1]My interest in this subject was triggered by Elliott Antokoletz's *The Music of Béla Bartók* (University of California Press, 1984), which in turn makes several references to George Perle's "Symmetrical Formations in the String Quartets of Béla Bartók" (*Music Review* no. 16 [November 1955], pp 300-312).

[2]In this article, "set" denotes a collection considered without regard to any particular order of its elements. This is the usual mathematical sense of the word, as opposed to a sense sometimes used in musical theory, according to which "set" is equivalent to "row".

[3]corresponding to the notes within an octave on a piano.

or of the form

$\langle u \rangle$, specifying u steps clockwise from the first (12 o'clock) tetrachord element
to the second,
again u steps from the second to the third,
and $6 - u$ steps from the third to the fourth,
(so again $6 - u$ steps from the fourth on again to the first).
(1.2)

It is to be understood that the type of a tetrachord is independent of its rotational

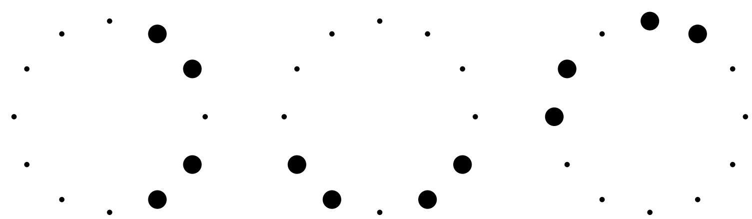


Figure 2: Three PS tetrachords of type $\langle 1, 2 \rangle$.

position. For example, the PS tetrachords of Figure 2 are all of the same type, namely $\langle 1, 2 \rangle$.

Figure 1 may be accepted as an (extensional) *definition* of inversional pitch-class symmetry for tetrachords in the twelve-pitch-class context: the PS tetrachords being exactly the pitch-class sets of the types shown. However, Figure 1 also suggests, and I hereby adopt, a more general *intensional* definition which is equivalent in this case:

a pitch-class set is PS if and only if it coincides with its own reflection
in a diameter of the pitch-class circle. (1.3)

For example, the tetrachord illustrating PS type $\langle 1, 2 \rangle$ in Figure 1 coincides with its own reflection in the 2 o'clock-8 o'clock diameter. For tetrachords, definition (1.3) is equivalent to defining one to be PS if and only if it and its reflection have the same type.

I trust that, with careful consideration, most readers will find:

- definition (1.3) for the relevant sort of symmetry to be clear and rigourous enough,

- the principles guiding the enumeration of Figure 1 to be intuitively apparent, and therefore
- the enumeration itself complete.[4]

An exact musical interpretation of the foregoing depends on how the twelve (clock) positions in each component of Figure 1 are taken to represent the conventional pitch classes. This can be done in several ways without affecting pitch-symmetry. The two most important are shown in Figure 3. Taking the integers $0, 1, \ldots, 11$ to specify pitch classes according to the number of semitones they lie nominally above an specified reference pitch class, the left-hand member of Figure 3 shows a circle of semitones; the right-hand member shows the familiar circle of perfect fifths. As suggested by the caption, each step on the left corresponds exactly to seven steps on the right and vice-versa.

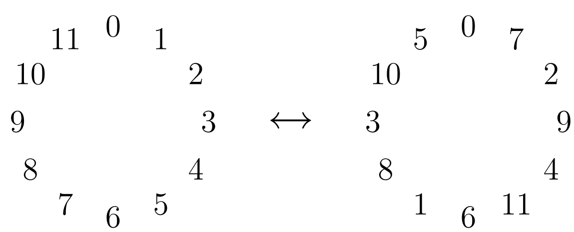


Figure 3: Inner automorphism $x \mapsto 7x \pmod{12}$ of $\mathbf{Z}_{12}$.

To say that the choice between the circle-of-semitones and circle-of-fifths interpretations does not affect pitch symmetry is merely to say that a given pitch-class set is either PS according to both interpretations (like, e.g. $\{0, 1, 2, 3\}$) or according to neither (like, e.g. $\{0, 1, 2, 4\}$). This does not imply that the type of a given PS tetrachord is the same in both interpretations or that a given PS type specifies the same pitch sets in both. In fact, $\{0, 1, 2, 3\}$, for example, is PS of type $\langle 1, 1\rangle$ in the circle of semitones, but of type $\langle 2, 3\rangle$ in the circle of fifths. Conversely, the PS tetrachord type $\langle 1, 1\rangle$ specifies semitone clusters in the circle-of-semitones interpretation, but pure quartal/quintal sonorities in the circle-of-fifths interpretation.

Tables 1 and 2 show some points of musical significance for all the (twelve-tone) PS tetrachord types. Table 1 is based on the interpretation of the symbols $\langle 1, 1\rangle$,

[4] To see that the forms (1.1) and (1.2) exhaust the possibilities, consider that a PS 4-chord must have an even number of elements (hence 0 or 2) on the reflecting diameter.

symbol*	Perle number	Forte number	interval vector	equivalent**
$\langle 1,1\rangle$	3	4-1	$\langle 3,2,1,0,0,0\rangle$	$\langle 2,3\rangle$
$\langle 1,2\rangle$	8	4-3	$\langle 2,1,2,1,0,0\rangle$	$\langle 3,2\rangle$
$\langle 1,3\rangle$	9	4-7	$\langle 2,0,1,2,1,0\rangle$	$\langle 4,1\rangle$
$\langle 1,4\rangle$	10	4-8	$\langle 2,0,0,1,2,1\rangle$	$\langle 1,4\rangle$
$\langle 1,5\rangle$	2	4-9	$\langle 2,0,0,0,2,2\rangle$	$\langle 1,5\rangle$
$\langle 2,1\rangle$	4	4-10	$\langle 1,2,2,0,1,0\rangle$	$\langle 2,1\rangle$
$\langle 2,2\rangle$	11	4-21	$\langle 0,3,0,2,0,1\rangle$	$\langle 2,2\rangle$
$\langle 2,3\rangle$	5	4-23	$\langle 0,2,1,0,3,0\rangle$	$\langle 1,1\rangle$
$\langle 2,4\rangle$	7	4-25	$\langle 0,2,0,2,0,2\rangle$	$\langle 2,4\rangle$
$\langle 3,1\rangle$	12	4-17	$\langle 1,0,2,2,1,0\rangle$	$\langle 3,1\rangle$
$\langle 3,2\rangle$	13	4-26	$\langle 0,1,2,1,2,0\rangle$	$\langle 1,2\rangle$
$\langle 3,3\rangle$	1	4-28	$\langle 0,0,4,0,0,2\rangle$	$\langle 3,3\rangle$
$\langle 4,1\rangle$	14	4-20	$\langle 1,0,1,2,2,0\rangle$	$\langle 1,3\rangle$
$\langle 1\rangle$	6	4-6	$\langle 2,1,0,0,2,1\rangle$	$\langle 1\rangle$
$\langle 2\rangle$	15	4-24	$\langle 0,2,0,3,0,1\rangle$	$\langle 2\rangle$

* circle-of-semitones interpretation

** circle-of-fifths interpretation

Perle number (as “four-note collection”): see George Perle, *Serial Composition and Atonality,* University of California Press, (6th ed., rev.) 1991, pp 149f.

Forte number: see Allen Forte, *The Structure of Atonal Music,* Yale University Press, 1973.

interval vector: numbers of intervals within the tetrachord comprising respectively 1, 2, 3, 4, 5, and 6 semitones.

Table 1: PS tetrachord types, further details (1).

symbol*	acoustic root	content	descriptive name	equivalent**
$\langle 1, 1\rangle$	0 7 2	{M2, P5, M6} {M2, P4, P5} {P4, P5, m7}	0Q	$\langle 2, 3\rangle$
$\langle 1, 2\rangle$	9 0	{m3, P5, m7} {M3, P5, M6}	9N	$\langle 3, 2\rangle$
$\langle 1, 3\rangle$	0 4	{M3, P5, M7} {m3, P5, m6}	0M	$\langle 4, 1\rangle$
$\langle 1, 4\rangle$	0 11	{A4, P5, M7} {m2, P5, m6}	0T	$\langle 1, 4\rangle$
$\langle 1, 5\rangle$	0, 6	{A1, A4, P5}	0X, 6X	$\langle 1, 5\rangle$
$\langle 2, 1\rangle$	2	{P5, M6, m7}	2V	$\langle 2, 1\rangle$
$\langle 2, 2\rangle$	0 2	{M2, M3, A4} {M2, M3, m7}	0W	$\langle 2, 2\rangle$
$\langle 2, 3\rangle$	11	{m2, M2, m3}	11S	$\langle 1, 1\rangle$
$\langle 2, 4\rangle$	2, 8	{M3, d5, m7}	2F, 8F	$\langle 2, 4\rangle$
$\langle 3, 1\rangle$	9	{m3, M3, P5}	9L	$\langle 3, 1\rangle$
$\langle 3, 2\rangle$	8	{m2, m3, d4}	8K	$\langle 1, 2\rangle$
$\langle 3, 3\rangle$	0, 3, 6, 9	{m3, d5, d7}	0D, 3D, 6D, 9D	$\langle 3, 3\rangle$
$\langle 4, 1\rangle$	4	{P5, m6, M7}	4U	$\langle 1, 3\rangle$
$\langle 1\rangle$	0 7	{m2, M2, P5} {P4, d5, P5}	0R	$\langle 1\rangle$
$\langle 2\rangle$	0 4 8	{M2, M3, m6} {d4, m6, m7} {M3, A4, A5}	8A	$\langle 2\rangle$

* circle-of-fifths interpretation

** circle-of-semitones interpretation

acoustic root: nominally lower pitch class of a P5, M3, or m3 interval within the tetrachord.

content: relative to the indicated acoustic root as P1.

descriptive name: numbers may conveniently be replaced by usual pitch-class names—C, C♯, etc.

Table 2: PS tetrachord types, further details (2).

etc with reference to the circle of semitones; Table 2 with reference to the circle of fifths. The descriptive names in Table 2 are ones I use as chord or function symbols in analytical phases of my compositional work.

2 Generalization

The foregoing generalizes readily (if a bit fussily), as follows.

Let n and k be integers, $2 \leq k \leq n$. Then the sequence of intervals of a *PS k-chord* in an n-step scale has one of the following three forms:

$$u_1 \ldots, u_L, v, u_L \ldots, u_1, w, \text{ where } k \text{ is even, } k \geq 4\,, \tag{2.1}$$
$$L = \frac{k}{2} - 1\,,$$
$$w = n - (2\sum_{i=1}^{L} u_i + v),$$
$$\text{(without loss of generality, } v \leq w)$$

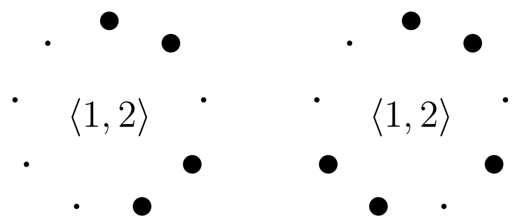
symbol: $\langle u_1 \ldots, u_L, v \rangle$ (a $\frac{k}{2}$-tuple);

$$u_1 \ldots, u_L, u_L \ldots, u_1, w, \text{ where } k \text{ is odd, } k \geq 3\,, \tag{2.2}$$
$$L = \frac{k-1}{2},$$
$$w = n - 2\sum_{i=1}^{L} u_i$$

symbol: $\langle u_1 \ldots, u_L \rangle$ (a $\frac{k-1}{2}$-tuple);

$$u_1 \ldots, u_L, u_L \ldots, u_1, \text{ where } n \text{ and } k \text{ are both even, } k \geq 2\,, \tag{2.3}$$
$$L = \frac{k}{2},$$
$$u_L = \frac{n}{2} - \sum_{i=1}^{L-1} u_i\,,$$
$$\text{(without loss of generality, } u_1 \leq u_L)$$

symbol: $\langle u_1 \ldots, u_{L-1} \rangle$ (a $(\frac{k}{2} - 1)$-tuple).



$k = 4$, form (2.1) $k = 5$, form (2.2)

Figure 4: Example PS 4- and 5-chord types, $n = 9$.

Figure 4 shows two examples in a nine-pitch-class context. Note that interpretation of the $\langle \ldots \rangle$ symbols depends on k. This is true for all n, but is not especially bothersome; in practice one is normally interested in particular values of k as well as n.

3 PS Hexachords in a Twelve-Tone Scale

Figure 5 shows all the PS 6-chord *(hexachord)* types in a twelve-pitch-class context ($k = 6$, $n = 12$). Necessarily, the complement[5] of a set of any of these types is also a PS hexachord. For six of the types, the complements are of the same type as the originals; these types, which may be called *self-complementary,* are marked (*).

Because PS hexachords characterize a class of tone rows important for twelve-tone serial composition (see page 13), I now digress briefly to that subject.

A *tone row* is a particular ordering of the twelve conventional pitch classes. A tone row is typically used along with one or more of its other *transformations:* its retrograde, its inversion, its retrograde inversion, or transpositions of any of these or of itself. A tone row is said to be *combinatorial* (or *semi-combinatorial*)[6] if the first six pitch classes of one or more of its transformations other than its retrograde[7] are complementary to the first six pitch classes of its original form. The following

[5] I.e. the pitch classes (among the twelve) which are not members of the set.
[6] These terms may be further qualified as *"hexachordally"* or *"by hexachords"*.
[7] Trivially, any row combines in the indicated way with its own retrograde.

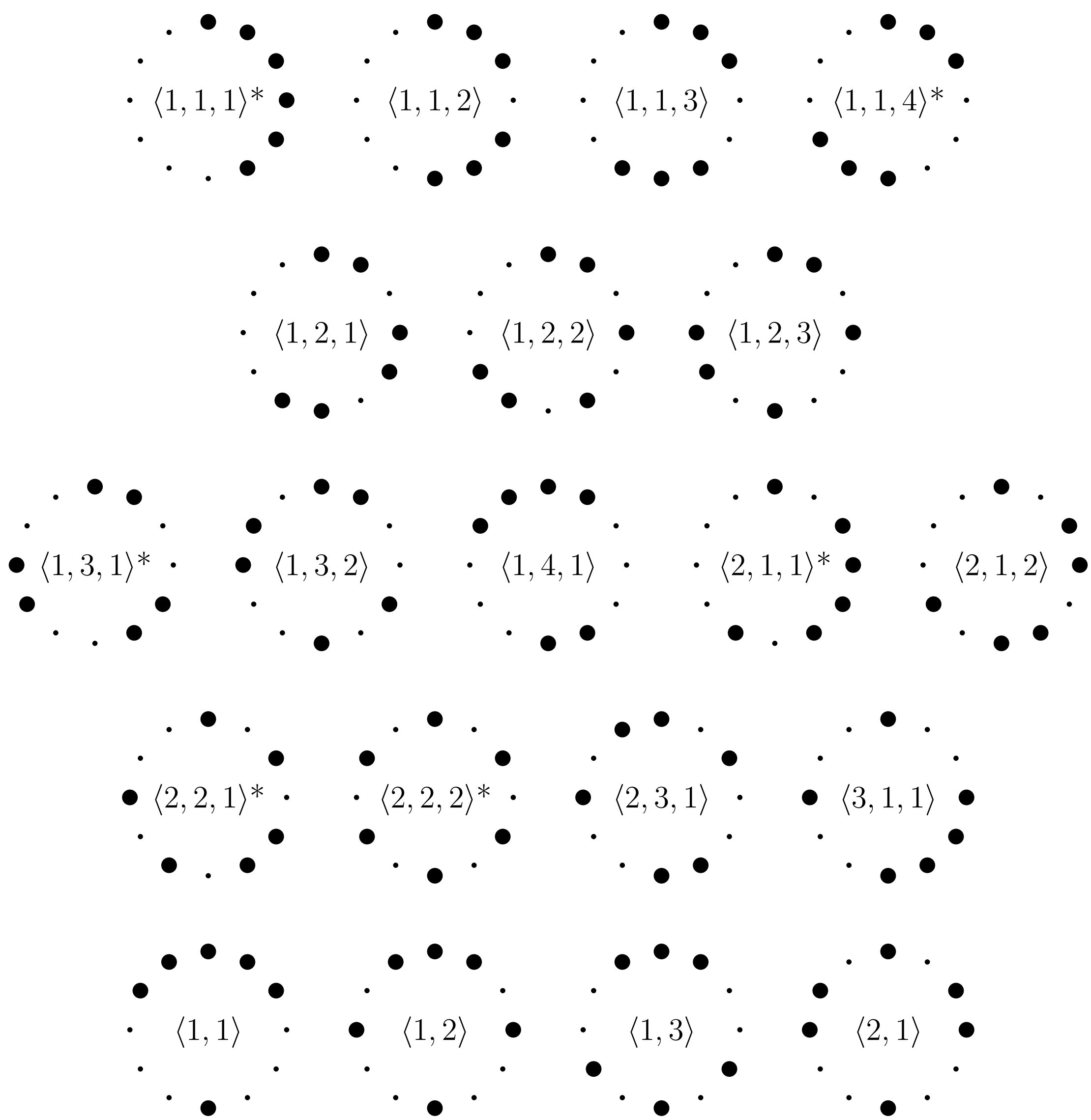


Figure 5: PS 6-chord types, $n = 12$. (* indicates self-complementary.)

				combinatoriality		
symbol*	Perle number	Forte number	interval vector	T	I	RI
$\langle 1,1,1\rangle$	4	6-1	$\langle 5,4,3,2,1,0\rangle$	•	•	•
$\langle 1,1,2\rangle$	7B	6-Z4	$\langle 4,3,2,3,2,1\rangle$			•
$\langle 1,1,3\rangle$	8B	6-Z6	$\langle 4,2,1,2,4,2\rangle$			•
$\langle 1,1,4\rangle$	3	6-7	$\langle 4,2,0,2,4,3\rangle$	•	•	•
$\langle 1,2,1\rangle$	9A	6-Z13	$\langle 3,2,4,2,2,2\rangle$			•
$\langle 1,2,2\rangle$	10B	6-Z26	$\langle 2,3,2,3,4,1\rangle$			•
$\langle 1,2,3\rangle$	11B	6-Z29	$\langle 2,2,4,2,3,2\rangle$			•
$\langle 1,3,1\rangle$	2	6-20	$\langle 3,0,3,6,3,0\rangle$	•	•	•
$\langle 1,3,2\rangle$	13B	6-Z49	$\langle 2,2,4,3,2,2\rangle$			•
$\langle 1,4,1\rangle$	8A	6-Z38	$\langle 4,2,1,2,4,2\rangle$			•
$\langle 2,1,1\rangle$	5	6-8	$\langle 3,4,3,2,3,0\rangle$	•	•	•
$\langle 2,1,2\rangle$	12B	6-Z23	$\langle 2,3,4,2,2,2\rangle$			•
$\langle 2,2,1\rangle$	6	6-32	$\langle 1,4,3,2,5,0\rangle$	•	•	•
$\langle 2,2,2\rangle$	1	6-35	$\langle 0,6,0,6,0,3\rangle$	•	•	•
$\langle 2,3,1\rangle$	11A	6-Z50	$\langle 2,2,4,2,3,2\rangle$			•
$\langle 3,1,1\rangle$	9B	6-Z42	$\langle 3,2,4,2,2,2\rangle$			•
$\langle 1,1\rangle$	7A	6-Z37	$\langle 4,3,2,3,2,1\rangle$			•
$\langle 1,2\rangle$	12A	6-Z45	$\langle 2,3,4,2,2,2\rangle$			•
$\langle 1,3\rangle$	10A	6-Z48	$\langle 2,3,2,3,4,1\rangle$			•
$\langle 2,1\rangle$	13A	6-Z28	$\langle 2,2,4,3,2,2\rangle$			•

* circle-of-semitones interpretation

Perle number (as "six-note collection"): see George Perle, *Serial Composition and Atonality,* University of California Press, (6th ed., rev.) 1991, pp 153-155. Note that 7A and 7B, 8A and 8B, etc are complementary types.

Forte number: see Allen Forte, *The Structure of Atonal Music,* Yale University Press, 1973.

interval vector: numbers of intervals within the hexachord comprising respectively 1, 2, 3, 4, 5, and 6 semitones.

combinatoriality: see text, pages 9f.

Table 3: PS hexachord types, further details.

examples illustrate this.[8]

$$\begin{aligned} P_0 &= \langle 0,1,3,4,5,8, \quad 2,6,7,9,10,11 \rangle \\ P_6 &= \langle 6,7,9,10,11,2, \quad 8,0,1,3,4,5 \rangle \end{aligned} \tag{3.1}$$

$$\begin{aligned} P_0 &= \langle 0,4,3,2,1,6, \quad 5,7,8,9,10,11 \rangle \\ I_{11} &= \langle 11,7,8,9,10,5, \quad 6,4,3,2,1,0 \rangle \end{aligned} \tag{3.2}$$

$$\begin{aligned} P_0 &= \langle 0,1,2,4,5,6, \quad 3,7,8,9,10,11 \rangle \\ RI_6 &= \langle 7,8,9,10,11,3, \quad 0,1,2,4,5,6 \rangle \end{aligned} \tag{3.3}$$

These rows are said to be respectively *(semi-)combinatorial by transposition* (3.1), *by inversion* (3.2), and *by retrograde inversion* (3.3). The musical reason for using such combinatorial pairs together is to allow for passages which retain the feature of non-repetition of pitch classes (generally characteristic of twelve-tone music) while varying internal content such as interval successions.

A tone row's combinatorial character in the above sense is clearly determined solely by the unordered hexachord comprising its first six pitch classes—in fact, solely by that hexachord's type (in the sense indicated at the beginning of this article, whereby all transpositions of a given hexachord are of the same type). The following facts are evident or readily deducible from the relevant definitions.

- A tone row is combinatorial by transposition if and only if the hexachord type of its first six pitch classes is self-complementary.
- A tone row is combinatorial by inversion if and only if the hexachord comprising the complement of its first six pitch classes (equivalently, the hexachord comprising its last six pitch classes) is an inverse[9] of the hexachord comprising its first six pitch classes.
- A tone row is combinatorial by retrograde inversion if and only if the hexachord comprising its first six pitch classes is PS.
- A tone row satisfying any two of the above three criteria satisfies all three. Such tone rows are said to be *fully combinatorial* or *all-combinatorial.*

[8]Various notations for the transformations of a row are in use. That used here is typical.

[9]Two pitch class sets are said to be *inverses* of one another if their members can be paired so that all pairs have the same sum *modulo* 12. E.g. the two rows in example (3.2), where the pairs all sum to 11.

We have reached the objective of this digression. The PS hexachord types (shown in Figure 5 and Table 3) are exactly those corresponding to tone rows combinatorial by retrograde inversion. The PS hexachord types which are also self-complementary, marked (*), correspond to fully combinatorial tone rows. For example (a row with first six pitch classes a hexachord of type $\langle 2,2,1\rangle$):

$$\begin{aligned}
\mathrm{P}_0 &= \langle 0,2,4,5,7,9, \quad 1,3,6,8,10,11\rangle\\
\mathrm{P}_6 &= \langle 6,8,10,11,1,3, \quad 7,9,0,2,4,5\rangle\\
\mathrm{I}_3 &= \langle 3,1,11,10,8,6, \quad 2,0,9,7,5,4\rangle\\
\mathrm{RI}_9 &= \langle 10,11,1,3,6,8, \quad 0,2,4,5,7,9\rangle
\end{aligned}$$

All twelve-tone hexachord types, both combinatorial and noncombinatorial, are enumerated in “Hauer’s Tropes and the Enumeration of Twelve-Tone Hexachords” on this website.

4 A Formal Treatment

This section provides a formal justification for the assertion made above (page 8) that the sequence of intervals of a *PS k-chord* in an n-step scale has one of the three forms (2.1)-(2.3). In the end, this boils down to subscript-chasing and may well therefore be no more compelling than the bald assertion, which seems to me intuitively hard to doubt. Still, in the words of Dostoevsky,[10] since it is already written, let it stand.

4.1 Nomenclature

The universe of discourse here is $\mathbf{Z}_n$, the cyclic group with elements $\{0\ \ldots, n-1\}$.[11] Here $\mathbf{Z}_n$ is considered only as an additive group; multiplication in $\mathbf{Z}_n$ is not relevant. Elements of $\mathbf{Z}_n$ are referred to as *pitch-classes* or *intervals.* As usual, for $P \subseteq \mathbf{Z}_n$, $t \in \mathbf{Z}_n$, $P+t = \{p+t \mid p \in P\}$, $-P = \{-p \mid p \in P\}$.

For $0 \le k \le n$, define a *k-chord* to be a set of k distinct pitch classes (i.e. distinct elements of $\mathbf{Z}_n$). For $1 \le k \le n$, define a *k-chord type* to be a set of $k-1$ distinct non-zero intervals (i.e. distinct non-zero elements of $\mathbf{Z}_n$). For a given k-chord type

$$T = \{i_1\ \ldots, i_{k-1}\}, \tag{4.1}$$

[10] *The Brothers Karamazov,* “From the Author”.

[11] I.e. the integers modulo n. For example, in $\mathbf{Z}_{12}$, $7+7=2$, $7+5=0$, $5-7=10$, $-8=4$.

say that a set of pitch-classes $C \subseteq \mathbf{Z}_n$ *realizes* T if and only if there is some $r \in \mathbf{Z}_n$ such that

$$C = \{r, r + i_1 \ \ldots, r + i_{k-1}\}. \tag{4.2}$$

Necessarily, such a set C is a k-chord. Together, (4.1) and (4.2) specify the nomenclature used here: A k-chord type T comprises the set of $k-1$ intervals from a particular pitch class r in a realization C to all the other pitch classes of C. For example, the 3-chord type $\{4, 7\}$ in $\mathbf{Z}_{12}$ (major third, perfect fifth) is realized by both $\{0, 4, 7\}$ and $\{3, 7, 10\}$.[12] Trivially, the unique 1-chord type $\varnothing$ is realized by $\{r\}$ for any $r \in \mathbf{Z}_n$.

Since the negative of a non-zero element of $\mathbf{Z}_n$ is non-zero, $-T$ is a k-chord type if and only if T is. Furthermore, $-C$ realizes $-T$ if and only if C realizes T.

For further discussion of k-chord types, it is convenient to define the *completion* $\widehat{T}$ of a k-chord type T as the set obtained by adjoining 0 to T:

$$\widehat{T} = T \cup \{0\}.$$

With this definition, (4.1) and (4.2) may be more compactly related by saying that a set of pitch-classes $C \subseteq \mathbf{Z}_n$ realizes a k-chord T if and only if there is some $r \in \mathbf{Z}_n$ such that

$$C = \widehat{T} + r.^{13} \tag{4.3}$$

In particular, $\widehat{T} = \widehat{T} + 0$ is a realization of T.

Obviously $\widehat{-T} = -\widehat{T}$.

4.2 Equivalence of k-chord types

Distinct k-chord types as defined here are not necessarily functionally different. For example, as well as $\{4, 7\}$, the 3-chord type $\{3, 8\}$ (minor third, minor sixth) is also realized by both $\{0, 4, 7\}$ and $\{3, 7, 10\}$ in $\mathbf{Z}_{12}$. For k-chord types T and U, say T is *equivalent* to U ($T \equiv U$) if and only if they are realized by the exactly the same subsets (k-chords) of $\mathbf{Z}_n$.[14] Obviously, $\equiv$ is an equivalence relation in the usual mathematical sense for k-chord types in $\mathbf{Z}_n$.

[12] As usual, the order in which set elements are written is insignificant; $\{7, 3, 10\} = \{10, 3, 7\}$, etc.

[13] Of course, I could have included 0 in all k-chord types by definition, making the notion of completion unnecessary, but I prefer to make the sufficiency of $k-1$ intervals to express a k-chord type as obvious as possible.

[14] The notion of k-chord type defined here does not distinguish among chord inversions in the musical sense.

It is easy to see that

$$T \equiv U \text{ if and only if } \widehat{U} = \widehat{T} + r \text{ for some } r \in \mathbf{Z}_n. \tag{4.4}$$

For example, in $\mathbf{Z}_{12}$: $\widehat{\{4,7\}} = \{0,4,7\} = \{8,0,3\}+4 = \widehat{\{3,8\}}+4$, and $\{3,8\} \equiv \{4,7\}$.

Since $0 \in \widehat{U}$, it is also easy to see that

$$T \equiv U \text{ if and only if } \widehat{U} = \widehat{T} - i \text{ for some } i \in \widehat{T}. \tag{4.5}$$

It follows that the $\equiv$-equivalence class of a given k-chord type has at most k members; that is, that a given k-chord type is equivalent to at most $k-1$ other k-chord types. For example, in $\mathbf{Z}_{12}$: $\{0,4,7\} - 4 = \{8,0,3\}$, $\{0,4,7\} - 7 = \{5,9,0\}$, and the set of 3-chord types equivalent to $\{4,7\}$ comprises exactly $\{4,7\}$, $\{3,8\}$, and $\{5,9\}$. A k-chord type can have fewer than k $\equiv$-equivalents. For example, the 4-chord type $\{3,6,9\}$ in $\mathbf{Z}_{12}$ is $\equiv$-equivalent only to itself.

It is also easy to see that

$$-T \equiv -U \text{ if and only if } T \equiv U. \tag{4.6}$$

4.3 Pitch-Class Symmetry

Say that $C \subseteq \mathbf{Z}_n$ is *pitch-symmetric*[15] *(PS)* if and only if

$$-C = C + r \text{ for some } r \in \mathbf{Z}_n.^{16} \tag{4.7}$$

For example, in $\mathbf{Z}_{12}$: $\{0,3,7,10\}$ (minor seventh chord) is PS since $-\{0,3,7,10\} = \{0,9,5,2\} = \{0,3,7,10\} + 2$. On the other hand, $\{0,3,7\}$ (minor triad) is not PS, as the only forms of $\{0,3,7\} + r$ containing 0 are $\{0,3,7\}$, $\{0,4,9\}$, and $\{0,5,8\}$, whereas $-\{0,3,7\} = \{0,5,9\}$. Any set of two or fewer elements of $\mathbf{Z}_n$ is PS.

One would expect this notion of pitch symmetry to extend naturally to k-chord types, and it does. Suppose C and D both realize some k-chord type T and that C is PS. Then, applying (4.3) and (4.7), it is easy to see that D is also PS. It follows that either all or none of the realizations of a given k-chord type are PS.

[15] As in Sections 1-3 above, the symmetry considered here can be visualised as reflection in a diameter of a circular arrangement of n points representing the elements of $\mathbf{Z}_n$. This corresponds to pitch-class inversion, but not in the special sense the term "inversion" has with respect to chords in music—hence my avoidance of the term "inversion". (See previous footnote.)

[16] The "$+r$" terms here and in (4.3) and (4.4), corresponding to musical transposition, are properly visualized as rotation in the circular representation of $\mathbf{Z}_n$.

Define a k-chord type T to be *pitch-symmetric (PS)* if and only if a realization of T is pitch-symmetric. By the above, this is equivalent to requiring that *every* realization of T be PS. From (4.3)-(4.7), a k-chord type T is PS if and only if

$$-T \equiv T. \tag{4.8}$$

4.4 Characterization of PS k-Chord Types

It may not be completely obvious how (4.7) or (4.8) capture the idea of pitch symmetry in k-chords, and there are one or two subtleties. This section provides all details.

Let T be a PS k-chord type and let

$$\widehat{T} = \{i_0, i_1 \ldots, i_{k-1}\}, \text{ with } 0 = i_0 < i_1 \ldots < i_{k-1} < n \text{ in } \mathbf{Z}.$$[17]

Then

$$-\widehat{T} = \{i_0, n - i_{k-1} \ldots, n - i_1\} \text{ and } 0 = i_0 < n - i_{k-1} \ldots < n - i_1 < n\,. \tag{4.9}$$

Since T is PS, it is also true (by (4.5), (4.7), and (4.8)) that, for some m with $0 \leq m < k$,

$$-\widehat{T} = \{(i_0 - i_m) \bmod n, (i_1 - i_m) \bmod n \ldots, (i_{k-1} - i_m) \bmod n\}.^{18} \tag{4.10}$$

For example, let $n = 24, k = 8$, and let X be the PS type $\{3, 5, 6, 8, 11, 15, 20\}$ as in Figure 6. Then

$$\begin{aligned}\widehat{X} = \{0, 3, 5, 6, 8, 11, 15, 20\} \text{ and } -\widehat{X} &= \{0, 4, 9, 13, 16, 18, 19, 21\}\\ &= \{13, 16, 18, 19, 21, 0, 4, 9\}\\ &= \{(0-11) \bmod n \ldots, (20-11) \bmod n\};\end{aligned}$$

in this case $m = 5, i_m = 11$.

(For this subsection, let $\overline{x}$ denote $x \bmod k$.) The elements listed in (4.10) increase from $n + (i_0 - i_m)$ to $n + (i_{\overline{m-1}} - i_m)$ and, if $m \neq 0$, from $0 = i_m - i_m$ to $i_{k-1} - i_m$ (in

[17] Throughout this section, $i_0, i_1 \ldots, i_{k-1}$ are mainly considered as elements of $\mathbf{Z}_n$, where $<$ is not defined. In this subsection, however, "$<$" refers to their relation in $\mathbf{Z}$; i.e. simply as integers, albeit elements of $\{0, 1 \ldots, n-1\}$. (Subscripts are always treated as elements of $\mathbf{Z}$.)

[18] For $y \neq 0, x \bmod y = x - y \cdot \lfloor x/y \rfloor$, where $\lfloor u \rfloor$ denotes the largest integer $\leq u$.

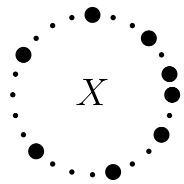


Figure 6: Example PS 8-chord type, $n = 24$.

the example, from 13 to 21 and from 0 to 9). Also $i_{k-1} - i_m < n - i_m = n + (i_0 - i_m)$. Accordingly, in ascending order, the terms of (4.10) are:

$$-\widehat{T} = \{0 = i_m - i_m \ldots, i_{k-1} - i_m, n + (i_0 - i_m) \ldots, n + (i_{m-1} - i_m)\}.^{19} \tag{4.11}$$

The elements of $-\widehat{T}$ must be equal term by term as written in (4.9) and (4.11). To make good use of this fact, for $0 \le j < k$, define

$$z_j = i_{\overline{j+1}} - i_j \quad \text{(i.e. } z_0 = i_1 - i_0, z_1 = i_2 - i_1 \ldots, z_{k-1} = i_0 - i_{k-1}).$$

These z_j are the intervals (differences) between adjoining pitch classes in the enumerations (4.9) and (4.11). Now the interval between the first two pitch classes in (4.9) is

$$(-i_{k-1}) - 0 = i_0 - i_{k-1} = z_{k-1},$$

and the interval between the first two pitch classes in (4.11) is

$$(i_{\overline{m+1}} - i_m) - (i_m - i_m) = i_{\overline{m+1}} - i_m = z_m,$$

since (4.9) and (4.11) are identically arranged, $z_{k-1} = z_m$. Proceeding thus through all the adjoining pairs of pitch classes in (4.9) and (4.11), one finds

$$\begin{aligned} z_{k-1} &= z_m \\ &\vdots \\ z_m &= z_{k-1} \\ z_{m-1} &= z_0 \\ &\vdots \\ z_0 &= z_{m-1}\,, \end{aligned} \tag{4.12}$$

[19] If $m = 0$, the second part of this enumeration is empty.

where z-subscripts on the left (from (4.9)) decrease from $k-1$ to 0, and z-subscripts on the right (from (4.11)) increase from m to $k-1$ and from 0 to $m-1$.[20] (In the example,

$$\begin{aligned}
z_7 &= z_5 = 4\\
z_6 &= z_6 = 5\\
z_5 &= z_7 = 4\\
z_4 &= z_0 = 3\\
z_3 &= z_1 = 2\\
z_2 &= z_2 = 1\\
z_1 &= z_3 = 2\\
z_0 &= z_4 = 3\,.)
\end{aligned}$$

The general significance of (4.12) is that an ordered enumeration of the intervals between adjoining pitch classes of $\widehat{T}$ (and therefore of any realization of T) comprises a sequence of k intervals in which both the first $k-m$ and the last m are palindromic. Since these two palindromes meet end-to-end in the circle of pitch classes represented by $\mathbf{Z}_n$, it follows that the whole k-element sequence of intervals has the mirror symmetry we expect.

To work out in detail the significance of (4.12), it is necessary to consider cases. For this purpose, let $p = \frac{k+m-1}{2}$ and $q = \frac{m-1}{2}$. Note that $p - q = \frac{k}{2}$.

If k is even and m is odd (as in the example), then p and q are both integers, and the enumeration (4.12) includes two vacuous equations:

$$\begin{aligned}
z_p &= z_p\\
z_q &= z_q\,.
\end{aligned}$$

In this case, the sequences $\langle z_{\overline{p+1}} \ldots, z_{\overline{q-1}}\rangle$ and $\langle z_{\overline{q+1}} \ldots, z_{\overline{p-1}}\rangle$ (subscripts wrapping around from $k-1$ to 0 as necessary) are reversals of one another, and (4.12) corresponds exactly with the form (2.1), where

$$\langle u_1 \ldots, u_L, v, u_L \ldots, u_1, w\rangle = \langle z_{\overline{p+1}} \ldots, z_{\overline{q-1}}, z_q, z_{\overline{q+1}} \ldots, z_{\overline{p-1}}, z_p\rangle.$$

If k is odd, then exactly one of p and q is an integer. Suppose m is even; then p and $q \pm \frac{1}{2}$ are integers, and the enumeration (4.12) includes just one vacuous equation:

$$z_p = z_p\,.$$

[20] If $m = 0$, the second part of the enumeration (4.12) is empty.

In this case, the sequences $\langle z_{\overline{p+1}} \ldots, z_{\overline{q-\frac{1}{2}}}\rangle$ and $\langle z_{\overline{q+\frac{1}{2}}} \ldots, z_{\overline{p-1}}\rangle$ (subscripts wrapping around from $k-1$ to 0 as necessary) are reversals of one another, and (4.12) corresponds exactly with the form (2.2), where

$$\langle u_1 \ldots, u_L, u_L \ldots, u_1, w\rangle = \langle z_{\overline{p+1}} \ldots, z_{\overline{q-\frac{1}{2}}}, z_{\overline{q+\frac{1}{2}}} \ldots, z_{\overline{p-1}}, z_p\rangle.$$

Similarly if k and m are both odd.

If k and m are both even, then neither p nor q is an integer, and the enumeration (4.12) includes no vacuous equations. In this case, the sequences $\langle z_{\overline{p+\frac{1}{2}}} \ldots, z_{\overline{q-\frac{1}{2}}}\rangle$ and $\langle z_{\overline{q+\frac{1}{2}}} \ldots, z_{\overline{p-\frac{1}{2}}}\rangle$ (subscripts wrapping around from $k-1$ to 0 as necessary) are reversals of one another, and (4.12) corresponds exactly with the form (2.3), where

$$\langle u_1 \ldots, u_L, u_L \ldots, u_1\rangle = \langle z_{\overline{p+\frac{1}{2}}} \ldots, z_{\overline{q-\frac{1}{2}}}, z_{\overline{q+\frac{1}{2}}} \ldots, z_{\overline{p-\frac{1}{2}}}\rangle.$$

QED.